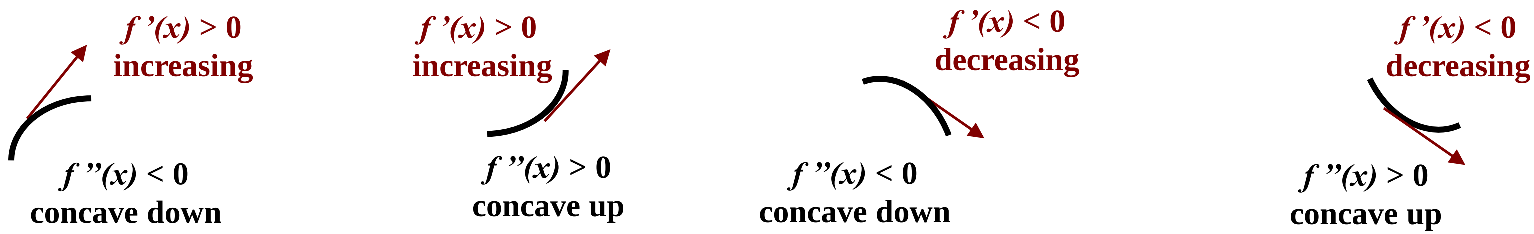
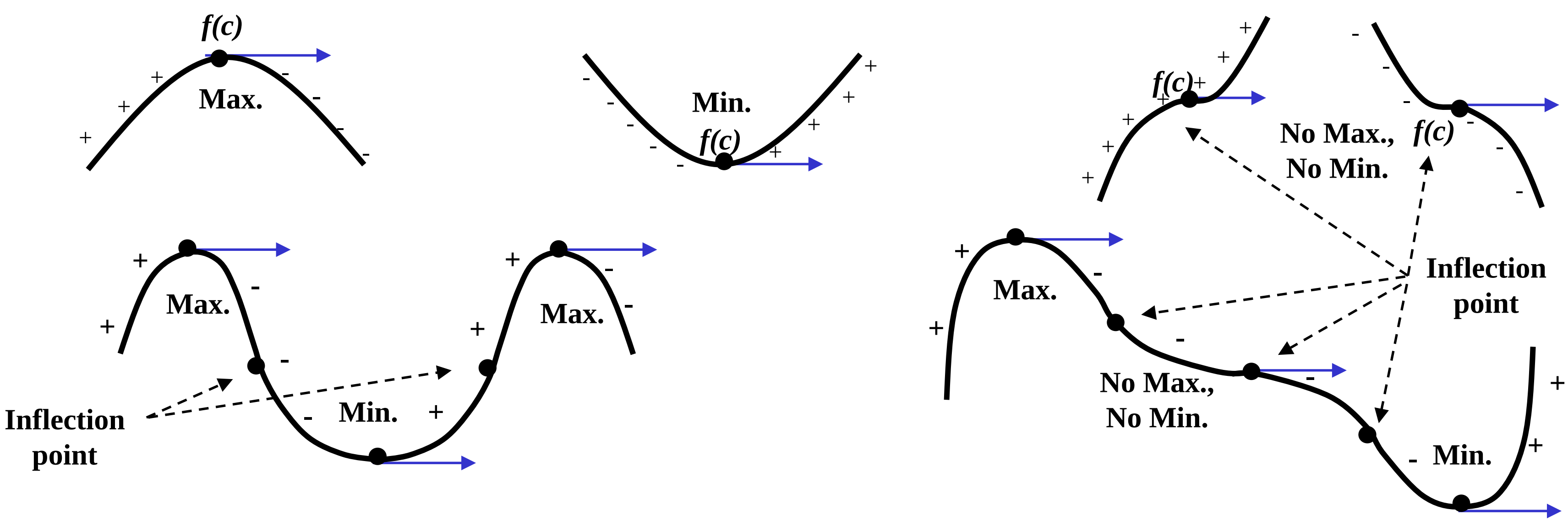
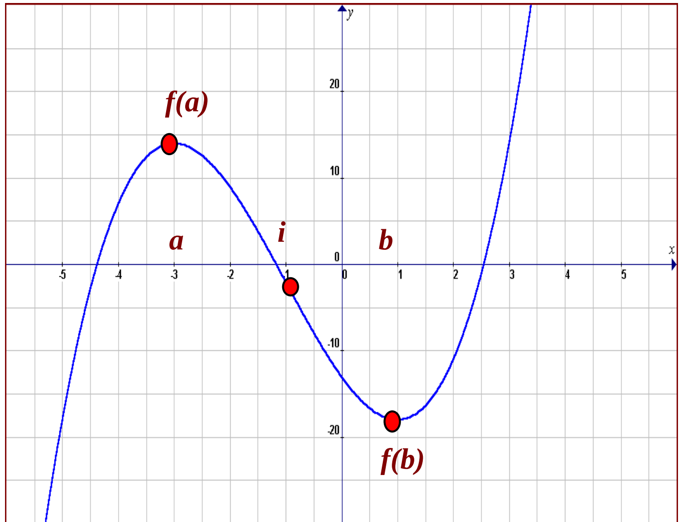
Application of Derivative in Analyzing the Properties of Functions

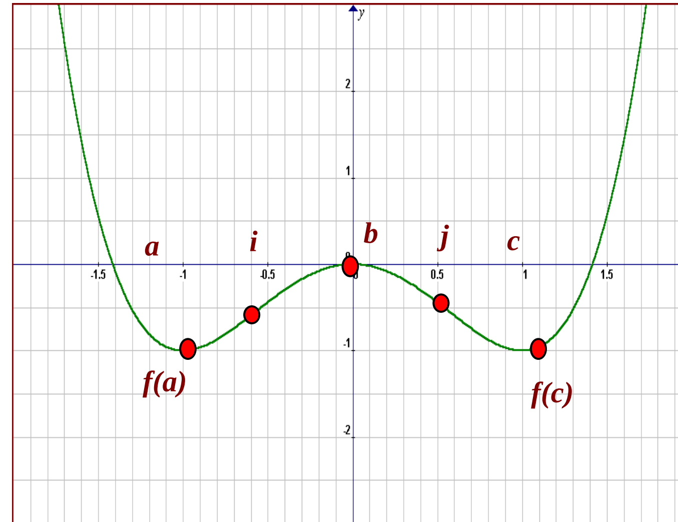
- $f'(x)$ indicates if the function is: **Increasing** when $f'(x) > 0$; **Decreasing** when $f'(x) < 0$
- $f''(x)$ indicates if the function is: **Concave up** when $f''(x) > 0$; **Concave down** when $f''(x) < 0$



- **Critical Point** c is where $f'(c) = 0$ *(tangent line is horizontal* → *)* ;
- **Inflection Point**: where $f''(x) = 0$ or where the function changes concavity, **No Min and no Max**
- If the sign of $f'(c)$ changes from + to - , then there is a **local Maximum**
- If the sign of $f'(c)$ changes from - to + , then there is a **local Minimum**
- If $f'(c) = 0$ but there is no sign change for $f'(c)$, then there is no local extreme, it is an **Inflection Point** (*concavity changes*)







Critical points, $f'(x) = 0$	**at :** $x = a$, $x = b$	**at :** $x = a$, $x = b$, $x = c$
Increasing, $f'(x) > 0$	**in :** $x < a$ **and** $x > b$	**in :** $a < x < b$ **and** $x > c$
Decreasing, $f'(x) < 0$	**in :** $a < x < b$	**in :** $x < a$ **and** $b < x < c$
Local Maximum	**at :** $x = a$, **Max** $= f(a)$	**at :** $x = b$, **Max** $= f(b)$
Local Minimum	**at :** $x = b$, **Min** $= f(b)$	**at:** $x = a$, $x = c$, **Min** $=$ $f(a)$ **and** $f(c)$
Inflection point, $f''(x) = 0$	**at :** $x = i$	**at :** $x = i$, $x = j$
Concave up, $f''(x) > 0$	**in:** $x > i$	**in:** $x < i$ **and** $x > j$
Concave Down, $f''(x) < 0$	**in:** $x < i$	**in:** $i < x < j$

Example: Analyze the function $f(x) = 3x^5 - 20x^3$

I) Using the First Derivative:

• **Step 1**: Locate the **critical points** where the derivative is = 0:

$f'(x) = 15x^4 - 60x^2$

$f'(x) = 0$ then $15x^2(x^2 - 4) = 0$.

Solve for x and you will find:

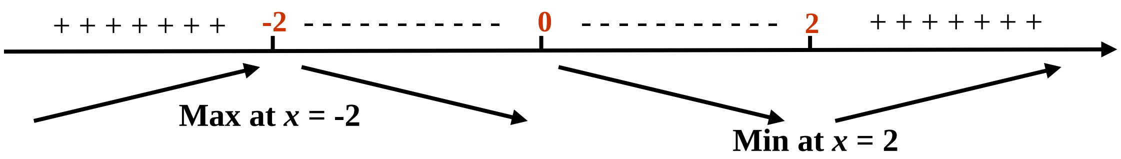
$\mathbf{x = -2}$, $\mathbf{x = 0}$ and $\mathbf{x = 2}$ as the critical points

• **Step 2:** Divide $f'(x)$ into intervals using the critical points found in the previous step, then choose a **test points:**

• **Step 3:** Find the derivative for the function in each test point:

$f'(x) = 15x^4 - 60x^2$	$f'(-3) = 675$	$f'(-1) = -45$	$f'(1) = -45$	$f'(3) = 675$
Sign	+ + + + + +	- - - - - - - - -	- - - - - - - - -	+ + + + + +
Shape	**Increasing**	**Decreasing**	**Decreasing**	**Increasing**
Intervals	$x < -2$	$-2 < x < 0$	$0 < x < 2$	$x > 2$

• **Step 4:** Look at both sides of each critical point:



Local Maximum at $x = -2$, Maximum = $f(-2) = 3(-2)^5 - 20(-2)^3 = 64$; or **Max (-2 , 64)**

Local Minimum at $x = 2$, Minimum = $f(2) = 3(2)^5 - 20(2)^3 = -64$; or **Min (2 , -64)**

II) Using the Second Derivative:

- **Step 5**: Locate the **inflection points** by making the second derivative is = 0:

We found $f'(x) = 15x^4 - 60x^2$ then $f''(x) = 60x^3 - 120x = 60x(x^2 - 2)$

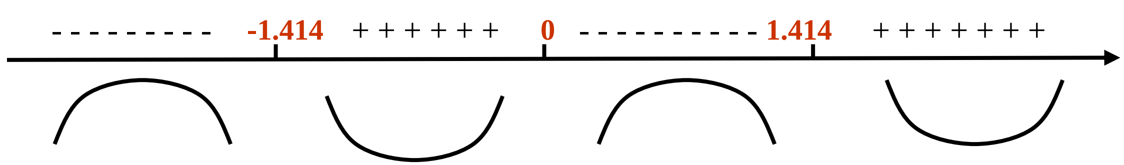
$f''(x) = 0$ then $60x(x^2 - 2) = 0$.

Solve for x and you will find $\boldsymbol{x = 0}$, $\boldsymbol{x =} \pm\sqrt{2} = -1.414, +1.414$

- **Step 6:** Divide $f''(x)$ into intervals using the inflection points found in the previous step, then choose a **test point:**

- **Step 7:** Find the second derivative for the function in each test point:

$\boldsymbol{f''(x) = 60x^3 - 120x}$	$f''(-2) = -$	$f''(-1) = +$	$f''(1) = -$	$f''(2) = +$
Sign	- - - - - - - - -	+ + + + + + +	- - - - - - - - -	+ + + + + +
Shape	**Concave Down**	**Concave Up**	**Concave Down**	**Concave Up**
Intervals	$\boldsymbol{x < -1.414}$	$\boldsymbol{-1.414 < x < 0}$	$\boldsymbol{0 < x < 1.414}$	$\boldsymbol{x > 1.414}$



- **Step 8**: Summarize all results in the following table:

Increasing in the intervals	***x* < -2 and *x* > 2**
Decreasing in the intervals	**-2 < *x* < 2**
Local Max. points and Max values:	Max. at *x* = -2 , **Max (-2 , 64)**
Local Min. points and Min values:	Min. at *x* = 2 , **Max (2 , -64)**
Inflection points at:	**(-1.414 , 39.6) , (0 , 0) , (-1.414 , -39.6)**
Concave Up in the intervals:	**-1.414 < *x* < 0** and ***x* > 1.414**
Concave Down in the intervals:	***x* < -1.414** and **0 < *x* < 1.414**

- **Step 9**: Sketch the graph: