MATH 211
Online Asynchronous Survey in Calculus and Analytical Geometry

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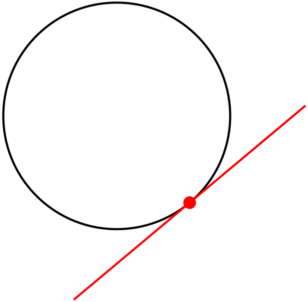
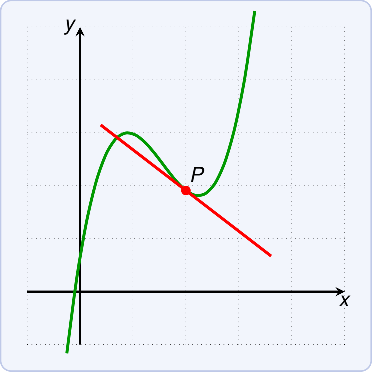
The Tangent

The **tangent** is a line that touches the curve:

- same direction as the curve at the point of contact

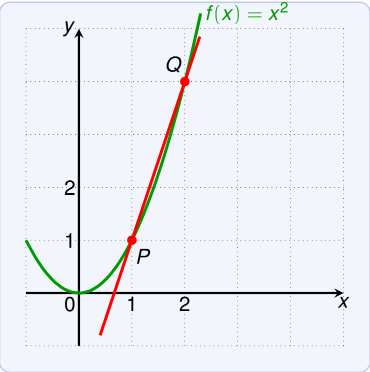




The Tangent: Example

Find the equation for the tangent to the curve x^2 at point $(1, 1)$.

- We need to know the slope m of x^2 at point $P = (1, 1)$.
- Take point $Q = (x, x^2)$ with $Q \neq P$ to compute the slope.



The slope from P to Q is:

$$m_{PQ} = \frac{Q_y - P_y}{Q_x - P_x} = \frac{x^2 - 1}{x - 1}$$

x	2			
m_{PQ}	3			

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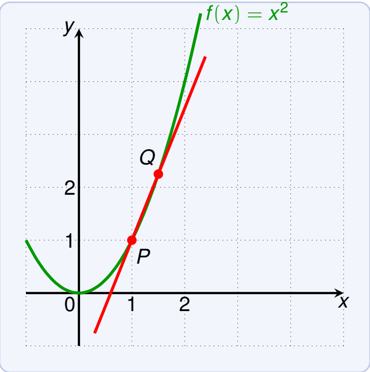

The slope from P to Q is:

$$m_{PQ} = \frac{Q_y - P_y}{Q_x - P_x} = \frac{x^2 - 1}{x - 1}$$

x	2	1.5		
m_{PQ}	3	2.5		

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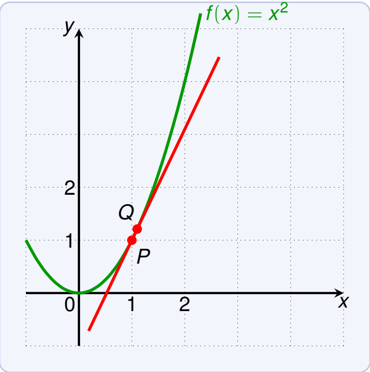

The slope from P to Q is:

$$m_{PQ} = \frac{Q_y - P_y}{Q_x - P_x} = \frac{x^2 - 1}{x - 1}$$

x	2	1.5	1.1	
m_{PQ}	3	2.5	2.1	

The Tangent: Example

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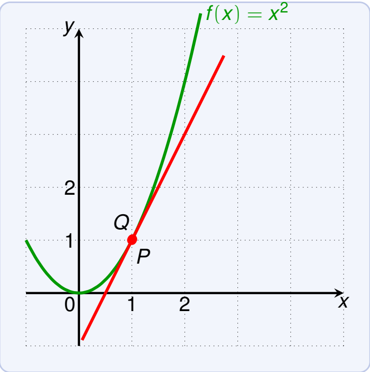


The slope from P to Q is:

$$m_{PQ} = \frac{Q_y - P_y}{Q_x - P_x} = \frac{x^2 - 1}{x - 1}$$

x	2	1.5	1.1	1.01
m_{PQ}	3	2.5	2.1	2.01

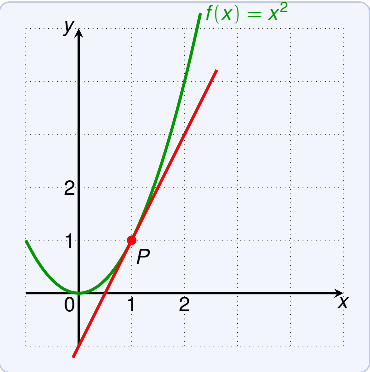
$\dots$

The closer Q to P, the closer m_{PQ} gets to 2.

The Tangent: Example

Find the equation for the tangent to the curve x^2 at point $(1, 1)$.

- We need to know the slope m of x^2 at point $P = (1, 1)$.
- Take point $Q = (x, x^2)$ with $Q \neq P$ to compute the slope.



The slope from P to Q is:

$$m_{PQ} = \frac{Q_y - P_y}{Q_x - P_x} = \frac{x^2 - 1}{x - 1}$$

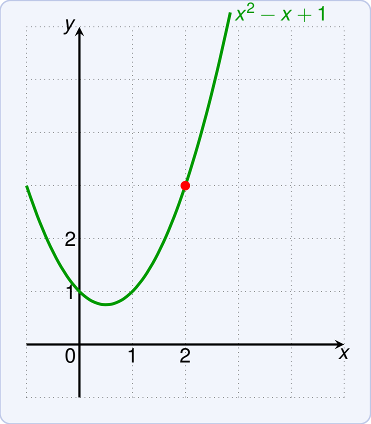
x	2	1.5	1.1	1.01	
m_{PQ}	3	2.5	2.1	2.01	$\dots$

The closer Q to P, the closer m_{PQ} gets to 2. Suggests that in P the slope $m = 2$.

Thus the tangent is $y - 1 = 2(x - 1)$ or $y = 2x - 1$.

The Limit of a Function

We investigate the function $x^2 - x + 1$ for values of x near 2.



from below ($x < 2$):

x	$f(x)$
1	1
1.5	1.75
1.9	2.71
1.99	2.9701
1.999	2.9970

from above ($x > 2$):

x	$f(x)$
2.5	4.75
2.2	3.64
2.1	3.31
2.01	3.0301
2.001	3.0030

From the tables we see: as x approaches 2, $f(x)$ approaches 3.

$$\lim_{x \to 2} (x^2 - x + 1) = 3$$

Limit: Definition

Suppose $f(x)$ is defined close to a (but not necessarily a itself). We write

$$\lim_{x\to a} f(x) = L$$

spoken: "the limit of $f(x)$, as x approaches a, is L"

if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a but not equal to a.

The values of $f(x)$ get closer to L as x gets closer to a.

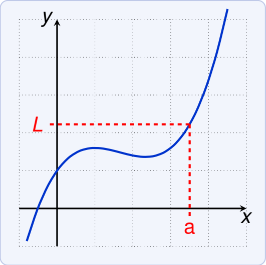
Alternative notation for $\lim_{x\to a} = L$:

$$f(x) \to L \quad \text{as} \quad x \to a$$

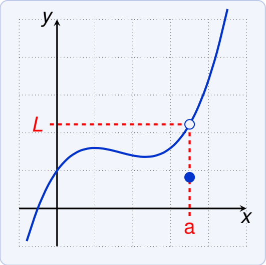
Limit: Continued

$\lim_{x \to a} f(x) = L$ if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a but not equal to a.

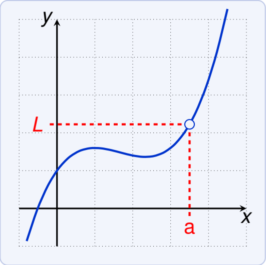
Note that we never consider $f(x)$ for $x = a$. The value of $f(a)$ does not matter. In fact, $f(x)$ need not be defined for $x = a$.



$f(a) = L$



$f(a) \neq L$



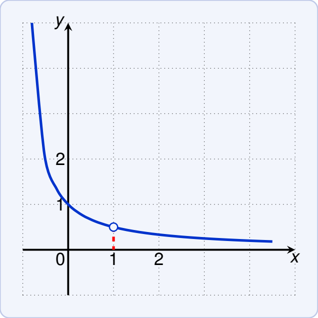
$f(a)$ undefined

In each of these cases we have $\lim_{x \to a} f(x) = L$!

Limit: Examples

Guess the value of

$$\lim_{x \to 1} \frac{x-1}{x^2-1}$$



The function is not defined at $x = 1$.
(does not matter for the limit)

from below:

x	$f(x)$
0.5	0.66667
0.9	0.52632
0.99	0.50251
0.999	0.50025

from above:

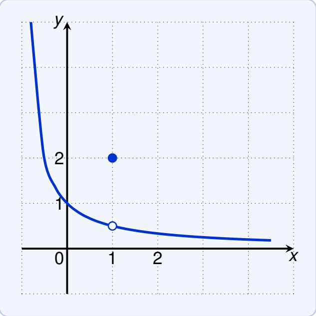
x	$f(x)$
1.5	0.40000
1.1	0.47619
1.01	0.49751
1.001	0.49975

From these values we guess that $\lim_{x \to 1} \frac{x-1}{x^2-1} = 0.5$.

Limit: Examples

Guess the value of $\lim_{x \to 1} g(x)$ where

$$g(x) = \begin{cases} \frac{x-1}{x^2-1} & \text{for } x \neq 1 \\ 2 & \text{for } x = 1 \end{cases}$$



As on the previous slide $\lim_{x \to 1} g(x) = 0.5$.
(recall that $g(1)$ does not matter for $\lim_{x \to 1} g(x)$).

Limit: Examples

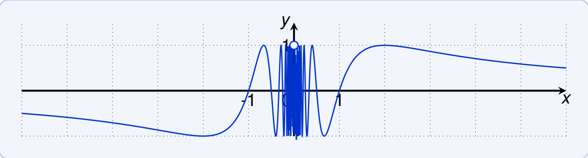
Guess the value of

$$\lim_{x \to 0} \sin \frac{\pi}{x}$$

x	$f(x)$
± 1	0
± 0.1	0
± 0.01	0
± 0.001	0

This suggest that the limit is 0.

However, this is wrong:



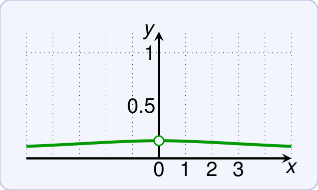
$\sin(\frac{\pi}{x}) = 0$ for arbitrarily small x, but also
$\sin(\frac{\pi}{x}) = 1$ for arbitrarily small x; e.g. $x = \frac{1}{2.5}, \frac{1}{4.5}, \frac{1}{6.5}, \ldots$

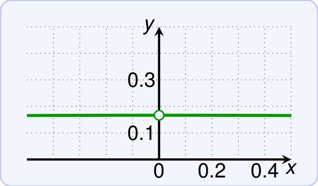
Hence: The limit $\lim_{x \to 0} \sin \frac{\pi}{x}$ does not exist.

Limit: Caution with Calculators

Guess the value of

$$\lim_{x \to 0} \frac{\sqrt{x^2+9}-3}{x^2}$$





x	$f(x)$
± 1.0	0.16228
± 0.5	0.16553
± 0.1	0.16662
± 0.01	0.16667
± 0.0001	0.20000
± 0.00001	0.00000
± 0.000001	0.00000

Is the limit 0? NO

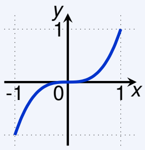
Problem: calculator gives wrong values!
For small x it rounds $\sqrt{x^2+9}-3$ to 0.

The correct limit is $\frac{1}{6} = 0.166666\ldots$

Limit: Examples

Guess the value of

$$\lim_{x\to 0}\left(x^3 + \frac{\cos 5x}{10000}\right)$$



x	$f(x)$
1	1.000028
0.5	0.124920
0.1	0.001088
0.01	0.000101

Looks like the limit is 0. But if we continue:

x	$f(x)$
0.005	0.00010009
0.001	0.00010000

We see actually that:

The value of the limit is 0.0001.

Limits and Calculators

Determining limits via calculators is a bad idea!

We have seen several sources of errors:

- we might stop too early, and draw wrong conclusions
- wrong results due to rounding in the calculator

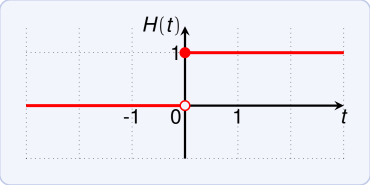
We need to compute limits precisely using limits laws...

Limit: Examples

The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

What is $\lim_{t \to 0} H(t)$?



- As t approaches 0 from the left, $H(t)$ approaches 0.
- As t approaches 0 from the right, $H(t)$ approaches 1.

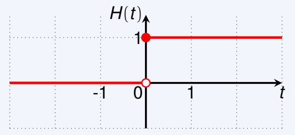
Thus there is not single number that $H(t)$ approaches.

The limit $\lim_{t \to 0} H(t)$ does not exist.

One-Sided Limits (From the Left)

The function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$



$H(t)$ approaches 0 as t approaches 0 from the left. We write:

$$\lim_{t \to 0^-} H(t) = 0$$

The symbol $t \to 0^-$ indicates that we consider only values $t < 0$.

We write $\lim_{x \to a^-} f(x) = L$ and say

"the left-hand limit of $f(x)$, as x approaches a, is L", *or*

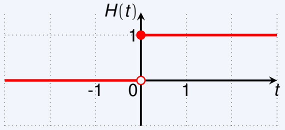
"the limit of $f(x)$, as x approaches a from the left, is L"

if we can make the values $f(x)$ arbitrarily close to L by taking x sufficiently close to a and $x < a$.

One-Sided Limits (From the Right)

The function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$



$H(t)$ approaches 1 as t approaches 0 from the right. We write:

$$\lim_{t \to 0^+} H(t) = 1$$

The symbol $t \to 0^+$ indicates that we consider only values $t > 0$.

We write $\lim_{x \to a^+} f(x) = L$ and say

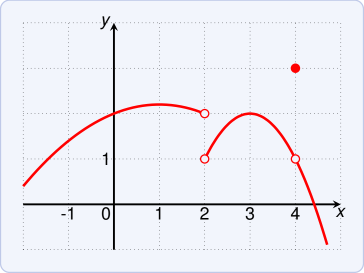
"the right-hand limit of $f(x)$, as x approaches a, is L", *or*

"the limit of $f(x)$, as x approaches a from the right, is L"

if we can make the values $f(x)$ arbitrarily close to L by taking x sufficiently close to a and $x > a$.

One-Sided Limits: Example

Consider the following graph of function $g(x)$:

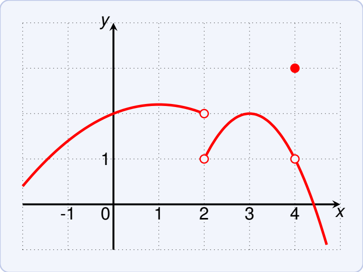


Use the graph to estimate the following values:

- $\lim_{x\to 2^-} = ?$ a 0 b 1 c 2 d 3 e does not exist

One-Sided Limits: Example

Consider the following graph of function $g(x)$:

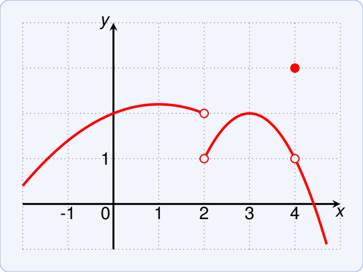


Use the graph to estimate the following values:

- $\lim_{x \to 2^-} = 2$
- $\lim_{x \to 2^+} = ?$ a 0 b 1 c 2 d 3 e does not exist

One-Sided Limits: Example

Consider the following graph of function $g(x)$:



Use the graph to estimate the following values:

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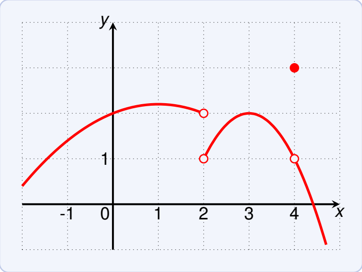
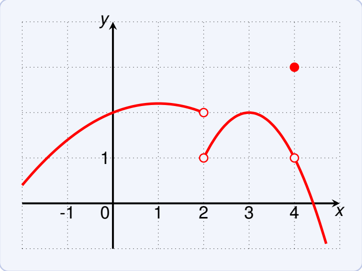


Use the graph to estimate the following values:

- $\lim_{x \to 2^-} = 2$
- $\lim_{x \to 2^+} = 1$
- $\lim_{x \to 2}$ does not exist
- $\lim_{x \to 4^-} = ?$ a 0 b 1 c 2 d 3 e does not exist

One-Sided Limits: Example

Consider the following graph of function $g(x)$:

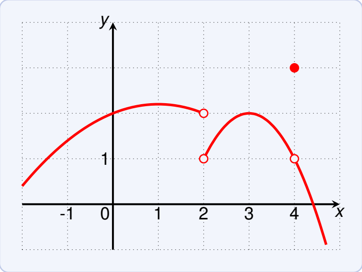


Use the graph to estimate the following values:

- $\lim_{x \to 2^-} = 2$
- $\lim_{x \to 2^+} = 1$
- $\lim_{x \to 2}$ does not exist
- $\lim_{x \to 4^-} = 1$
- $\lim_{x \to 4^+} = ?$ a 0 b 1 c 2 d 3 e does not exist

One-Sided Limits: Example

Consider the following graph of function $g(x)$:

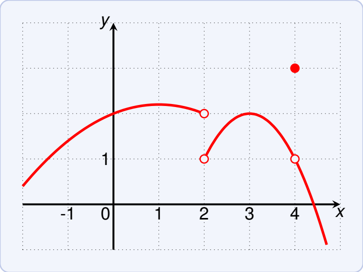


Use the graph to estimate the following values:

- $\lim_{x \to 2^-} = 2$
- $\lim_{x \to 2^+} = 1$
- $\lim_{x \to 2}$ does not exist
- $\lim_{x \to 4^-} = 1$
- $\lim_{x \to 4^+} = 1$
- $\lim_{x \to 4} = ?$ a 0 b 1 c 2 d 3 e does not exist

One-Sided Limits: Example

Consider the following graph of function $g(x)$:

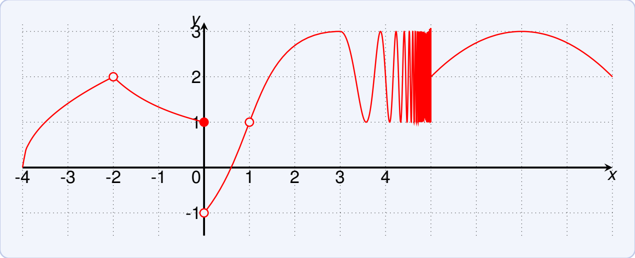


Use the graph to estimate the following values:

- $\lim_{x \to 2^-} = 2$
- $\lim_{x \to 2^+} = 1$
- $\lim_{x \to 2}$ does not exist
- $\lim_{x \to 4^-} = 1$
- $\lim_{x \to 4^+} = 1$
- $\lim_{x \to 4} = 1$

Infinite Limits: Example

Consider the following graph of function $g(x)$:

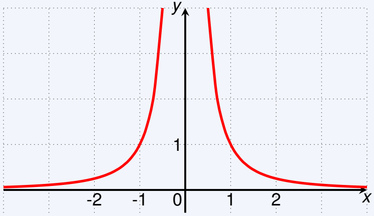


Use the graph to estimate the following values:

- $\lim_{x \to 3^-} g(x) = 3$
- $\lim_{x \to 3^+} g(x) = 3$
- $\lim_{x \to 3} g(x) = 3$
- $\lim_{x \to 1} g(x) = 1$
- $g(1) =$ undefined
- $g(0) = 1$
- $\lim_{x \to 0^-} g(x) = 1$
- $\lim_{x \to 0^+} g(x) = -1$
- $\lim_{x \to 0} g(x) =$ does not exist
- $\lim_{x \to 5^-} g(x) =$ does not exist
- $\lim_{x \to 5^+} g(x) = 2$
- $\lim_{x \to 5} g(x) =$ does not exist

Infinite Limits

We consider the function $\frac{1}{x^2}$. What is $\lim_{x\to 0} \frac{1}{x^2}$?



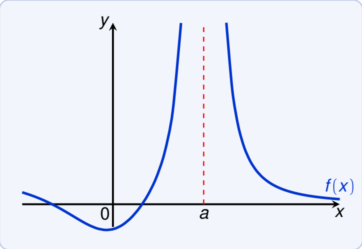
As x becomes close to 0, $\frac{1}{x^2}$ becomes very large. The values do not approach a number, so $\lim_{x\to 0} \frac{1}{x^2}$ does not exist!

Nevertheless, in this case, we write

$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

to indicate that the values become larger and larger.

Infinite Limits: Definition



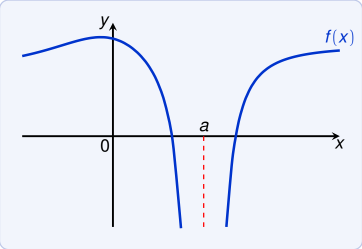
Suppose $f(x)$ is defined close to a (but not necessarily a itself). Then we write

$$\lim_{x \to a} f(x) = \infty$$

spoken: "the limit of $f(x)$, as x approaches a, is infinity"

if we can make the values of $f(x)$ arbitrarily large by taking x to be sufficiently close to a (but not equal to a).

Infinite Limits: Definition



Suppose $f(x)$ is defined close to a (but not necessarily a itself). Then we write

$$\lim_{x \to a} f(x) = -\infty$$

spoken: “the limit of $f(x)$, as x approaches a, is negative infinity”

if we can make the values of $f(x)$ arbitrarily large negative by taking x to be sufficiently close to a (but not equal to a).

Infinite One-Sided Limits

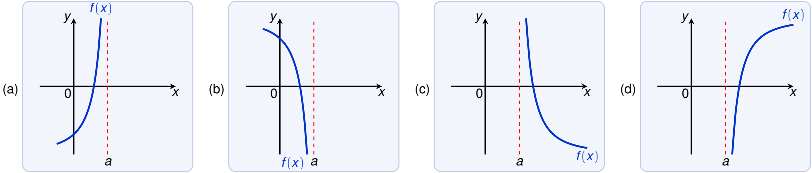
Like wise we define the one-sided infinite limits:

(a) $\lim_{x \to a^-} f(x) = \infty$

(b) $\lim_{x \to a^-} f(x) = -\infty$

(c) $\lim_{x \to a^+} f(x) = \infty$

(d) $\lim_{x \to a^+} f(x) = -\infty$



Note that ∞ and $-\infty$ are not considered numbers.

If $\lim_{x \to a} f(x) = \infty$ then $\lim_{x \to a} f(x)$ does not exist.
It indicates a certain way in which the limit does not exist.

Infinite Limits: Examples

Find

$$\lim_{x\to 3^-} \frac{2x}{x-3} \qquad \text{and} \qquad \lim_{x\to 3^+} \frac{2x}{x-3}$$

$\lim_{x\to 3^-} \frac{2x}{x-3} = ?$ a 0 b 1 c ∞ d $-\infty$

Infinite Limits: Examples

Find

$$\lim_{x \to 3^-} \frac{2x}{x-3} \qquad \text{and} \qquad \lim_{x \to 3^+} \frac{2x}{x-3}$$

$\lim_{x \to 3^-} \frac{2x}{x-3} = -\infty$

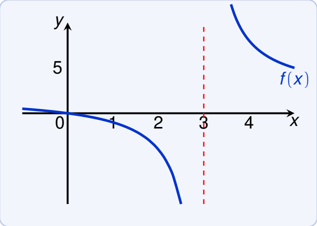
$\lim_{x \to 3^+} \frac{2x}{x-3} = ?$ a 0 b 1 c ∞ d $-\infty$

Infinite Limits: Examples

Find

$$\lim_{x\to 3^-} \frac{2x}{x-3} \qquad \text{and} \qquad \lim_{x\to 3^+} \frac{2x}{x-3}$$

$\lim_{x\to 3^-} \frac{2x}{x-3} = -\infty$
$\lim_{x\to 3^+} \frac{2x}{x-3} = \infty$



If x is close to 3 and $x < 3$ (approaching from the left), then:

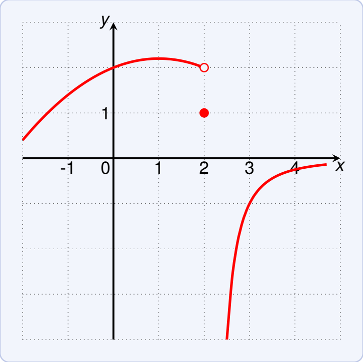
- $2x$ is close to 6,
- $x - 3$ is a small negative number,
- and thus $2x/(x-3)$ is a large negative number.

Hence $\lim_{x\to 3^-} \frac{2x}{x-3} = -\infty$.

Similarly for x close to 3 and $x > 3$, but now $x - 3$ is positive.

Infinite Limits: Example

Consider the following graph of function $g(x)$:

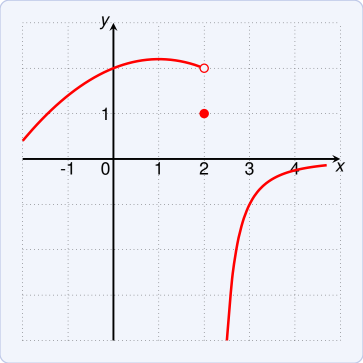


Use the graph to estimate the following values:

- $f(2) = ?$ (a) 0 (b) 1 (c) 2 (d) undefined

Infinite Limits: Example

Consider the following graph of function $g(x)$:

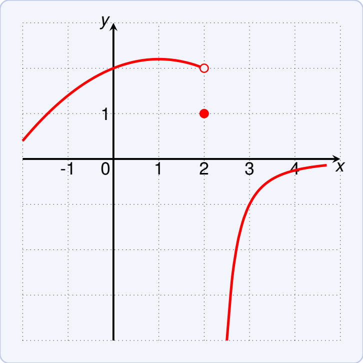


Use the graph to estimate the following values:

- $f(2) = 1$
- $\lim_{x\to 2^-} = ?$ (a) 1 (b) 2 (c) ∞ (d) $-\infty$ (e) does not exist

Infinite Limits: Example

Consider the following graph of function $g(x)$:

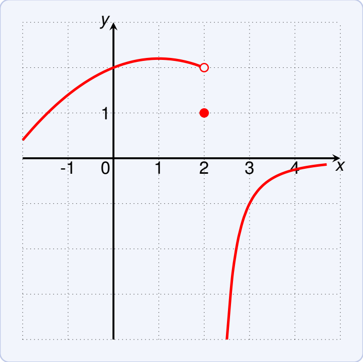


Use the graph to estimate the following values:

- $f(2) = 1$
- $\lim_{x\to 2^-} = 2$
- $\lim_{x\to 2^+} = ?$ a 1 b 2 c ∞ d $-\infty$ e does not exist

Infinite Limits: Example

Consider the following graph of function $g(x)$:

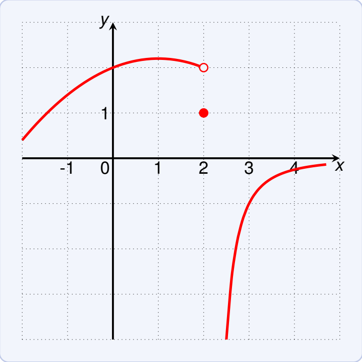


Use the graph to estimate the following values:

- $f(2) = 1$
- $\lim_{x \to 2^-} = 2$
- $\lim_{x \to 2^+} = -\infty$ (special case of ‘does not exist’)
- $\lim_{x \to 2} = ?$ a 1 b ∞ c $-\infty$ d does not exist

Infinite Limits: Example

Consider the following graph of function $g(x)$:



Use the graph to estimate the following values:

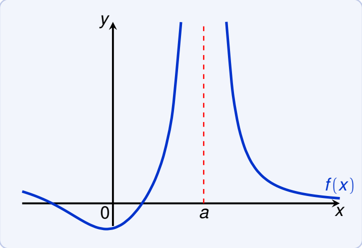
- $f(2) = 1$
- $\lim_{x \to 2^-} = 2$
- $\lim_{x \to 2^+} = -\infty$ (special case of 'does not exist')
- $\lim_{x \to 2}$ does not exist

Infinite Limits: Vertical Asymptotes

The line $x = a$ is a **vertical asymptote** of a function f if at least one of the following statements is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$

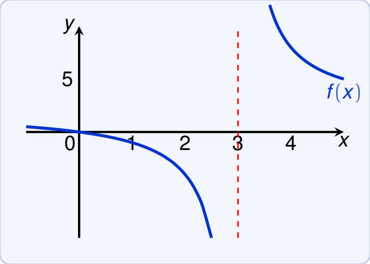
$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$



Infinite Limits: Vertical Asymptotes

What are the vertical asymptotes of

$$f(x) = \frac{2x}{x-3} \;\; ?$$



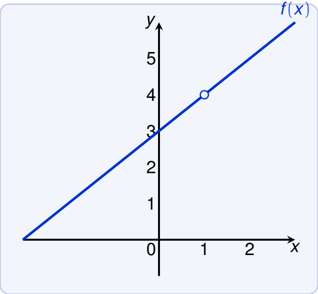
The function has the vertical asymptote $x = 3$:

$$\lim_{x \to 3^-} f(x) = -\infty$$

Infinite Limits: Vertical Asymptotes

What are the vertical asymptotes of

$$f(x) = \frac{x^2 + 2x - 3}{x - 1} \quad ?$$



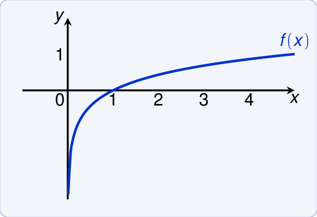
The function has no vertical asymptotes:

$$\frac{x^2 + 2x - 3}{x - 1} = (x + 3) \text{ for } x \neq 1$$

Infinite Limits: Vertical Asymptotes

What are the vertical asymptotes of

$$f(x) = \log_5 x \quad ?$$



The function has the vertical asymptote $x = 0$:

$$\lim_{x \to 0^+} \log_5 x = -\infty$$