MATH 211
Online Asynchronous Survey in Calculus and Analytical Geometry

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Calculating Limits using Limit Laws

We have seen that calculating limits with a calculator sometimes leads to incorrect results.

We will now see how to compute limits using **Limit Laws**:

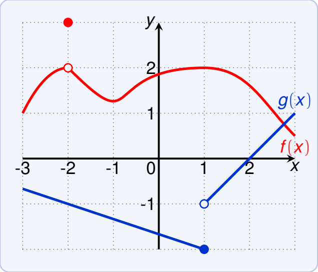
Let c be a constant, and let $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} [c \cdot f(x)] = c \cdot \lim_{x\to a} f(x)$
4. $\lim_{x\to a} [f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

These laws also work for one-sided limits $\lim_{x\to a^{\pm}}$.

Calculating Limits using Limit Laws

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a}[c\cdot f(x)] = c\cdot \lim_{x\to a} f(x)$
4. $\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$



Use these graphs to estimate:

1. $\lim_{x\to -2}[f(x)+5g(x)]$
$= \lim_{x\to -2} f(x) + \lim_{x\to -2}[5g(x)]$
$= \lim_{x\to -2} f(x) + 5\lim_{x\to -2} g(x)$
$= 2+5(-1)$
$= -3$

Calculating Limits using Limit Laws

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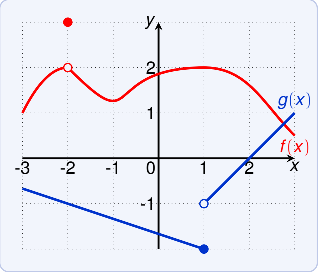

Use these graphs to estimate:

2. $\lim_{x\to 1} [f(x)g(x)]$
   $= \lim_{x\to 1} f(x) \cdot \lim_{x\to 1} g(x)$
   $\lim_{x\to 1} g(x)$ does not exist

   (we cannot use the limit laws)

Calculating Limits using Limit Laws

1. $\lim_{x\to a}\,[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
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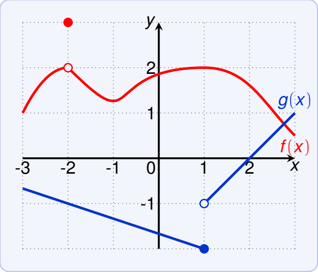

Use these graphs to estimate:

2a. $\lim_{x\to 1^-}\,[f(x)g(x)]$
$= \lim_{x\to 1^-} f(x)\cdot \lim_{x\to 1^-} g(x)$
$= 2\cdot -2 = -4$

2b. $\lim_{x\to 1^+}\,[f(x)g(x)]$
$= \lim_{x\to 1^+} f(x)\cdot \lim_{x\to 1^+} g(x)$
$= 2\cdot -1 = -2$

$\Longrightarrow$ $\lim_{x\to 1}\,[f(x)g(x)]$ does not exist

Calculating Limits using Limit Laws

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
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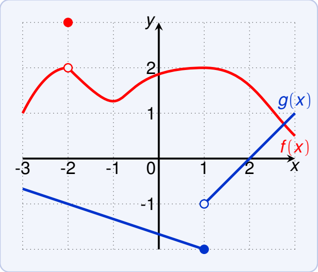

Use these graphs to estimate:

3. $\lim_{x\to 2} \frac{f(x)}{g(x)} = \frac{\lim_{x\to 2} f(x)}{\lim_{x\to 2} g(x)}$

   ↯ $\lim_{x\to 2} g(x) = 0$

   (we cannot use the limit laws)

Calculating Limits using Limit Laws

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
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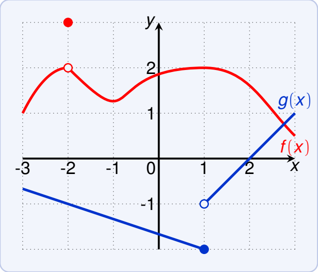

Use these graphs to estimate:

Lets try without limit laws:

3a. $\lim_{x\to 2^-} \frac{f(x)}{g(x)} = -\infty$
since $\lim_{x\to 2^-} f(x) \approx 1.6$, and $g(x)$ approaches 0, $g(x) < 0$

3b. $\lim_{x\to 2^+} \frac{f(x)}{g(x)} = \infty$
since $\lim_{x\to 2^+} f(x) \approx 1.6$, and $g(x)$ approaches 0, $g(x) > 0$

More Limits Laws

1. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} [c \cdot f(x)] = c \cdot \lim_{x\to a} f(x)$
4. $\lim_{x\to a} [f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a} [f(x)]^n = [\lim_{x\to a} f(x)]^n$ for n a positive integer
7. $\lim_{x\to a}\ c = c$
8. $\lim_{x\to a}\ x^n = a^n$
9. $\lim_{x\to a}\ \sqrt[n]{x} = \sqrt[n]{a}$ for n a positive integer
   (if n is even we require $a > 0$)
10. $\lim_{x\to a}\ \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ for n a positive integer
    (if n is even we require $\lim_{x\to a} f(x) > 0$)

Limit Laws: Examples

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
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    (if n is even we require $\lim_{x\to a} f(x) > 0$)

$$
\begin{aligned}
\lim_{x\to 5}(2x^2-3x+4) &= \lim_{x\to 5}(2x^2) - \lim_{x\to 5}(3x) + \lim_{x\to 5} 4 && \text{(law 1 and 2)}\\
&= 2\lim_{x\to 5}(x^2) - 3\lim_{x\to 5}(x) + 4 && \text{(law 3 and 7)}\\
&= 2\cdot 5^2 - 3\cdot 5 + 4 = 39 && \text{(law 8)}
\end{aligned}
$$

Limit Laws: Examples

$$
\begin{aligned}
&\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x} \\
&= \frac{\lim_{x\to-2}(x^3+2x^2-1)}{\lim_{x\to-2}(5-3x)} && \text{(law 5)} \\
&= \frac{\lim_{x\to-2} x^3 + 2\lim_{x\to-2} x^2 - \lim_{x\to-2} 1}{\lim_{x\to-2} 5 - 3\lim_{x\to-2} x} && \text{(law 1, 2, 3)} \\
&= \frac{(-2)^3 + 2\cdot(-2)^2 - 1}{5 - 3\cdot(-2)} && \text{(law 7, 8)} \\
&= -\frac{1}{11}
\end{aligned}
$$

Computing Limits: Direct Substitution Property

Direct Substitution Property

If f is a polynomial or a rational and a is in the domain of f, then:

$$\lim_{x \to a} f(x) = f(a)$$

Works also for one-sided limits $\lim_{x \to a^\pm} f(x) = f(a)$.
Works also for algebraic functions if $f(x)$ is defined close to a.

The function $f(x) = 2x^2 - 3x + 4$ is a polynomial and hence:

$$\lim_{x \to 5} f(x) = f(5) = 2 \cdot 5^2 - 3 \cdot 5 + 4 = 39$$

The function $g(x) = \frac{x^3 + 2x^2 - 1}{5 - 3x}$ is rational and -2 is in the domain; hence:

$$\lim_{x \to -2} g(x) = g(-2) = \frac{(-2)^3 + 2 \cdot (-2)^2 - 1}{5 - 3 \cdot (-2)} = -\frac{1}{11}$$

Computing Limits: Function Replacement

Function Replacement

If $f(x) = g(x)$ for all $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ (provided that the limit exists).

Actually it suffices $f(x) = g(x)$ when x is close to a.

Find $\lim_{x\to 1} \frac{x^2-1}{x-1}$.

- Direct substitution is not applicable because $x = 1$ is not in the domain.

We replace the function:

$$\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} \overset{\text{for } x \neq 1}{=} x+1$$

As a consequence

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} x+1 = 1+1 = 2$$

Computing Limits: Function Replacement

Find $\lim_{x \to 1} g(x)$ where

$$g(x) = \begin{cases} 2x + 1 & \text{for } x \neq 1, \\ \pi & \text{for } x = 1 \end{cases}$$

We have:

$$g(x) = 2x + 1 \quad \text{for all } x \neq 1$$

As a consequence:

$$\lim_{x \to 1} g(x) = \lim_{x \to 1} 2x + 1 = 2 + 1 = 3$$

Computing Limits: Function Replacement

Find

$$\lim_{h\to 0} \frac{(3+h)^2-9}{h}$$

We have:

$$\frac{(3+h)^2-9}{h} = \frac{9+6h+h^2-9}{h} = \frac{6h+h^2}{h} \overset{\text{for } h\neq 0}{=} 6+h$$

As a consequence:

$$\lim_{h\to 0} \frac{(3+h)^2-9}{h} = \lim_{h\to 0}(6+h) = 6$$

Computing Limits: Function Replacement

Find

$$\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$$

We have:

$$\frac{\sqrt{t^2+9}-3}{t^2} = \frac{\sqrt{t^2+9}-3}{t^2} \cdot \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3} = \frac{t^2+9-9}{t^2 \cdot (\sqrt{t^2+9}+3)}$$

$$= \frac{t^2}{t^2 \cdot (\sqrt{t^2+9}+3)} \overset{\text{for } t \neq 0}{=} \frac{1}{\sqrt{t^2+9}+3}$$

As a consequence:

$$\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \lim_{t\to 0} \frac{1}{\sqrt{t^2+9}+3}$$

$$= \frac{1}{\sqrt{\lim_{t\to 0}(t^2+9)}+3} \quad \text{by laws 5, 1, 9, 7}$$

$$= \frac{1}{\sqrt{9}+3} = \frac{1}{6}$$

Limits and One-Sided Limits

We recall the following theorem:

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

The theorem in words:

- The limit of $f(x)$, for x approaching a, is L if and only if the left-limit and the right-limit at a are both L.

The limit laws also apply for one-sided limits!

- if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$
  then $\lim_{x \to a} f(x)$ does not exist

Computing Limits: Function Replacement

Function replacement for one-sided limits:

If $f(x) = g(x)$ for all $x < a$, then $\lim_{x\to a^-} f(x) = \lim_{x\to a^-} g(x)$.

If $f(x) = g(x)$ for all $x > a$, then $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x)$.

Find $\lim_{x\to 2^-} g(x)$ where

$$g(x) = \begin{cases} x^2 & \text{for } x < 2 \\ 5x+1 & \text{for } x \geq 2 \end{cases}$$

We have

$$g(x) = x^2 \quad \text{for all } x < 2$$

Hence:

$$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} x^2 = 4$$

Computing Limits: Function Replacement

For one-sided limits we have:

If $f(x) = g(x)$ for all $x < a$, then $\lim_{x \to a^-} f(x) = \lim_{x \to a^-} g(x)$.

If $f(x) = g(x)$ for all $x > a$, then $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x)$.

Find $\lim_{x \to 0} |x|$ where

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

Since $|x| = x$ for all $x > 0$ we obtain:

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$$

Since $|x| = -x$ for all $x < 0$ we obtain:

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} -x = 0$$

Hence $\lim_{x \to 0} |x| = 0$.

Computing Limits: Function Replacement

For one-sided limits we have:

If $f(x) = g(x)$ for all $x < a$, then $\lim_{x\to a^-} f(x) = \lim_{x\to a^-} g(x)$.

If $f(x) = g(x)$ for all $x > a$, then $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x)$.

Proof that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

For all $x > 0$ we have $\frac{|x|}{x} = \frac{x}{x} = 1$. Thus

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} 1 = 1$$

For all $x < 0$ we have $\frac{|x|}{x} = \frac{-x}{x} = -1$. Thus

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} -1 = -1$$

Hence $\lim_{x\to 0} \frac{|x|}{x}$ does not exist since $\lim_{x\to 0^-} \frac{|x|}{x} \neq \lim_{x\to 0^+} \frac{|x|}{x}$.

Properties of Limits

If

- $f(x) \leq g(x)$ when x is near a (except possibly a),
- $\lim_{x \to a} f(x)$ exists, and
- $\lim_{x \to a} g(x)$ exist,

then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

Formally, near a means on $(a - \epsilon, a + \epsilon) \setminus \{a\}$ for some $\epsilon > 0$.

We have $x^3 \leq x^2$ for $x \in (-1, 1)$.

As a consequence:

$$\lim_{x \to a} x^3 \leq \lim_{x \to a} x^2$$

for all $a \in (-1, 1)$.

Properties of Limits

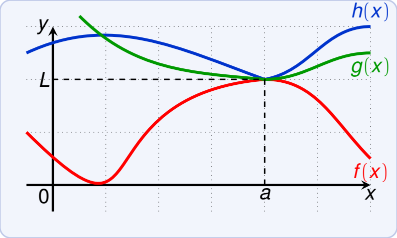
The Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly a) and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

then

$$\lim_{x \to a} g(x) = L$$



Here f is below g, and h is above g (close to a). If f and h have the same limit, then the squeezed function g also has.

Properties of Limits

Show that $\lim_{x \to 0} g(x) = 0$ where $g(x) = x^2 \cdot \sin \frac{1}{x}$.

The application of limit laws

$$\lim_{x \to 0} (x^2 \cdot \sin \frac{1}{x}) = (\lim_{x \to 0} x^2) \cdot (\lim_{x \to 0} \sin \frac{1}{x})$$

does not work since $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.

To apply the squeeze theorem we need:

- a function f smaller ($\leq$) than g, and
- a function h bigger ($\geq$) than g

for which $\lim_{x \to 0} f(x) = 0$ and $\lim_{x \to 0} h(x) = 0$.

We know that $-1 \leq \sin \frac{1}{x} \leq 1$ and hence

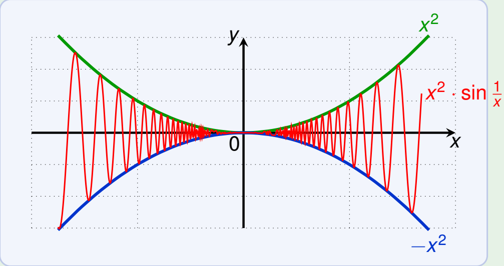
$$-x^2 \quad \leq \quad x^2 \cdot \sin \frac{1}{x} \quad \leq \quad x^2$$

Properties of Limits

We have

$$-x^2 \quad \leq \quad x^2 \cdot \sin\frac{1}{x} \quad \leq \quad x^2$$

We take $f(x) = -x^2$ and $h(x) = x^2$.



We know $\lim_{x\to 0} x^2 = 0$ and $\lim_{x\to 0} -x^2 = 0$.

Hence by the squeeze theorem we get: $\lim_{x\to 0} x^2 \cdot \sin\frac{1}{x} = 0$.