

MATH 211

Online Asynchronous Survey in Calculus and Analytical Geometry

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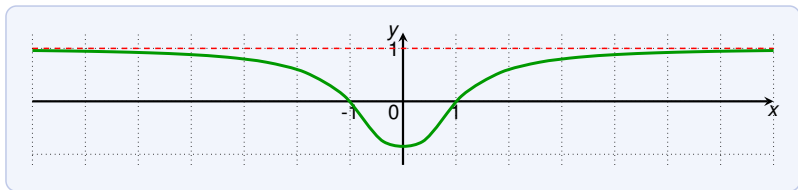


Limits at Infinity

Lets investigate the behavior of the function

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

when x becomes large:



As x grows larger, the values of $f(x)$ get closer and closer to 1.

This is expressed by

$$\lim_{x \rightarrow \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

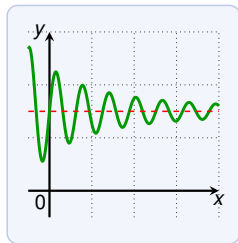
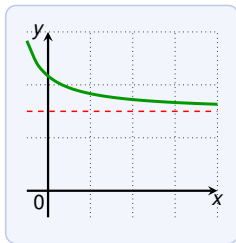
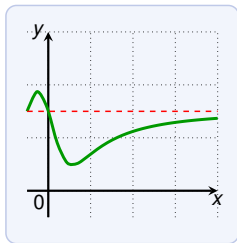
Limits at Infinity

Let f be a function defined on some interval (a, ∞) . Then

$$\lim_{x \rightarrow \infty} f(x) = L$$

spoken: “the limit of $f(x)$, as x approaches infinity, is L ”

if the values $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.



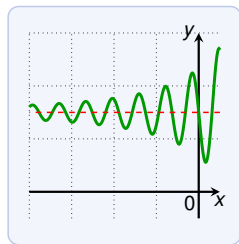
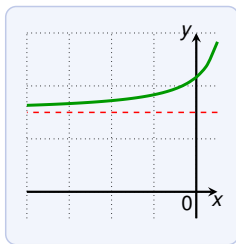
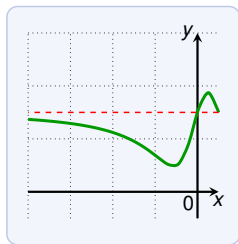
Limits at Infinity

Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \rightarrow -\infty} f(x) = L$$

spoken: “the limit of $f(x)$, as x approaches **negative** infinity, is L ”

if the values $f(x)$ can be made arbitrarily close to L by taking x sufficiently large **negative**.

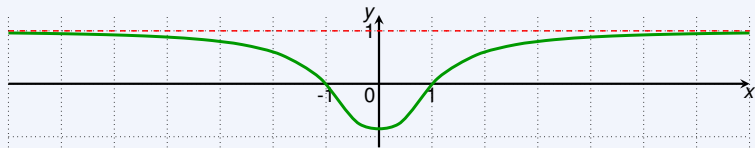


Limits at Infinity: Horizontal Asymptotes

The line $y = L$ is called **horizontal asymptote** of a function f if

$$\lim_{x \rightarrow \infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = L$$

The function $f(x) = \frac{x^2-1}{x^2+1}$ has a horizontal asymptote at $y = 1$.



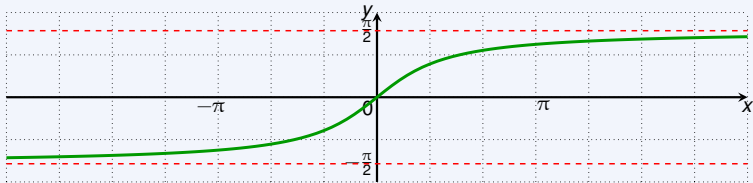
Limits at Infinity: Horizontal Asymptotes

The line $y = L$ is called **horizontal asymptote** of a function f if

$$\lim_{x \rightarrow \infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = L$$

The inverse tangent \tan^{-1} has horizontal asymptotes

$$y = -\frac{\pi}{2} \quad \text{and} \quad y = \frac{\pi}{2}$$



$$\lim_{x \rightarrow -\infty} \tan^{-1} x = -\frac{\pi}{2}$$

$$\lim_{x \rightarrow \infty} \tan^{-1} x = \frac{\pi}{2}$$

Limits at Infinity

Find $\lim_{x \rightarrow \infty} \frac{1}{x}$ and $\lim_{x \rightarrow -\infty} \frac{1}{x}$.

As x gets larger, $\frac{1}{x}$ gets closer to 0.

Thus $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$.

As x gets larger negative, $\frac{1}{x}$ gets closer to 0.

Thus $\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$.



The function has the horizontal asymptote $y = 0$.

Limits at Infinity: Laws

All **limits laws** for $\lim_{x \rightarrow a}$ work also for $\lim_{x \rightarrow \pm\infty}$, **except for:**

$$\lim_{x \rightarrow a} x^n = a^n$$

$$\lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a}$$

For example, we can derive the following important theorem:

For $r > 0$ we have

$$\lim_{x \rightarrow \infty} \frac{1}{x^r} = 0$$

and if x^r is defined for all x , then also

$$\lim_{x \rightarrow -\infty} \frac{1}{x^r} = 0$$

Proof

$$\lim_{x \rightarrow \infty} \frac{1}{x^r} = \lim_{x \rightarrow \infty} \left(\frac{1}{x}\right)^r = \left(\lim_{x \rightarrow \infty} \frac{1}{x}\right)^r = 0^r = 0$$

Limits at Infinity

Evaluate

$$\lim_{x \rightarrow \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

We have

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} &= \lim_{x \rightarrow \infty} \left(\frac{3x^2 - x - 2}{5x^2 + 4x + 1} \cdot \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} \right) \\ &= \lim_{x \rightarrow \infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}} \\ &= \frac{\lim_{x \rightarrow \infty} \left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{\lim_{x \rightarrow \infty} \left(5 + \frac{4}{x} + \frac{1}{x^2}\right)} \\ &= \frac{3}{5} \end{aligned}$$

Limits at Infinity

Evaluate

$$\lim_{x \rightarrow \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

We have

$$\lim_{x \rightarrow \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x \rightarrow \infty} \left(\frac{\sqrt{2x^2 + 1}}{3x - 5} \cdot \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} \right)$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{\sqrt{2x^2 + 1}}{x}}{3 - \frac{5}{x}} = \lim_{x \rightarrow \infty} \frac{\frac{\sqrt{2x^2 + 1}}{\sqrt{x^2}}}{3 - \frac{5}{x}} \quad \text{since } x > 0, x = \sqrt{x^2}$$

$$= \lim_{x \rightarrow \infty} \frac{\sqrt{2 + \frac{1}{x^2}}}{3 - \frac{5}{x}} = \frac{\lim_{x \rightarrow \infty} \sqrt{2 + \frac{1}{x^2}}}{\lim_{x \rightarrow \infty} \left(3 - \frac{5}{x}\right)}$$

$$= \frac{\sqrt{\lim_{x \rightarrow \infty} \left(2 + \frac{1}{x^2}\right)}}{3} = \frac{\sqrt{2}}{3}$$

Limits at Infinity

Evaluate

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

We have

$$\lim_{x \rightarrow \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x \rightarrow \infty} \left(\frac{\sqrt{2x^2 + 1}}{3x - 5} \cdot \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} \right)$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{\sqrt{2x^2 + 1}}{x}}{3 - \frac{5}{x}} = \lim_{x \rightarrow \infty} \frac{\frac{\sqrt{2x^2 + 1}}{\sqrt{x^2}}}{3 - \frac{5}{x}} \quad \text{since } x < 0, x = -\sqrt{x^2}$$

$$= \lim_{x \rightarrow \infty} \frac{\sqrt{2 + \frac{1}{x^2}}}{3 - \frac{5}{x}} = \frac{\lim_{x \rightarrow \infty} \sqrt{2 + \frac{1}{x^2}}}{\lim_{x \rightarrow \infty} \left(3 - \frac{5}{x}\right)}$$

$$= \frac{\sqrt{\lim_{x \rightarrow \infty} \left(2 + \frac{1}{x^2}\right)}}{3} = \frac{\sqrt{2}}{3}$$

Limits at Infinity

Evaluate

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

We have

$$\lim_{x \rightarrow \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x \rightarrow \infty} \left(\frac{\sqrt{2x^2 + 1}}{3x - 5} \cdot \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} \right)$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{\sqrt{2x^2 + 1}}{x}}{3 - \frac{5}{x}} = \lim_{x \rightarrow \infty} \frac{\frac{\sqrt{2x^2 + 1}}{-\sqrt{x^2}}}{3 - \frac{5}{x}} \quad \text{since } x < 0, x = -\sqrt{x^2}$$

$$= \lim_{x \rightarrow \infty} \frac{-\sqrt{2 + \frac{1}{x^2}}}{3 - \frac{5}{x}} = \frac{\lim_{x \rightarrow \infty} \sqrt{2 + \frac{1}{x^2}}}{\lim_{x \rightarrow \infty} (3 - \frac{5}{x})}$$

$$= \frac{-\sqrt{\lim_{x \rightarrow \infty} (2 + \frac{1}{x^2})}}{3} = \frac{-\sqrt{2}}{3}$$

Limits at Infinity

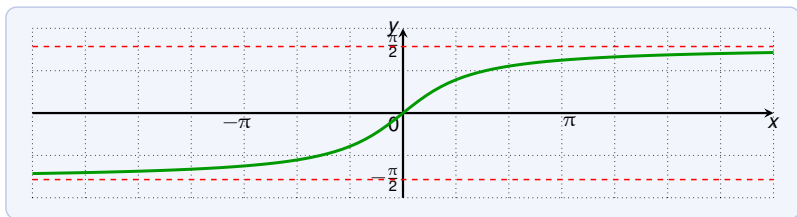
Evaluate

$$\lim_{x \rightarrow \infty} (\sqrt{x^2 - 1} - x)$$

We have

$$\begin{aligned}\lim_{x \rightarrow \infty} (\sqrt{x^2 - 1} - x) &= \lim_{x \rightarrow \infty} \left(\frac{\sqrt{x^2 - 1} - x}{1} \cdot \frac{\sqrt{x^2 - 1} + x}{\sqrt{x^2 - 1} + x} \right) \\ &= \lim_{x \rightarrow \infty} \frac{x^2 - 1 - x^2}{\sqrt{x^2 - 1} + x} \\ &= \lim_{x \rightarrow \infty} -\frac{1}{\sqrt{x^2 - 1} + x} \\ &= \lim_{x \rightarrow \infty} -\frac{1}{\sqrt{x^2 - 1} + x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\ &= \lim_{x \rightarrow \infty} -\frac{\frac{1}{x}}{\sqrt{1 - \frac{1}{x^2}} + 1} = \frac{0}{2} = 0\end{aligned}$$

Limits at Infinity



The graph of \tan^{-1} .

Evaluate

$$\lim_{x \rightarrow 2^+} \tan^{-1} \left(\frac{1}{x-2} \right) = \lim_{x \rightarrow \infty} \tan^{-1} x = \frac{\pi}{2}$$

Limits at Infinity

For exponential function we have:

$$\lim_{x \rightarrow \infty} a^x = 0 \quad \text{for } 0 \leq a < 1$$

$$\lim_{x \rightarrow -\infty} a^x = 0 \quad \text{for } a > 1$$

For any polynomial P and $a > 1$ we have

$$\lim_{x \rightarrow \infty} \frac{P(x)}{a^x} = 0$$

since the exponential function grows faster than any polynomial.

For any polynomial P and $0 < a < 1$ we have

$$\lim_{x \rightarrow -\infty} \frac{P(x)}{a^x} = 0$$

Limits at Infinity

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$$

A good **heuristic (this is not a law)** for to look at:

- ▶ the fastest growing addend of $f(x)$
- ▶ the fastest growing addend of $g(x)$

Typically, the other addends do not matter.

$$\lim_{x \rightarrow \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{3}{5}$$

$$\lim_{x \rightarrow \infty} \frac{\sqrt{5x^3 + 1} + 2x^2}{x^2 + 1} = 2$$

$$\lim_{x \rightarrow \infty} \frac{5x^3 + x + x \cdot x^2}{2x^3 - x} = 3$$

Limits at Infinity

Evaluate

$$\lim_{x \rightarrow \infty} \frac{3x^5 + x^2 - 2}{x^2 - x + 2^x} = 0$$

since 2^x grows faster than any polynomial.

Evaluate

$$\lim_{x \rightarrow \infty} \frac{3x^2 + x}{5x^2 - x + 5^{-x}} = \frac{3}{5}$$

since $\lim_{x \rightarrow \infty} 5^{-x} = 0$.

Evaluate

$$\lim_{x \rightarrow 0^-} e^{\frac{1}{x}} = \lim_{x \rightarrow -\infty} e^x = 0$$

Limits at Infinity

Evaluate

$$\lim_{x \rightarrow \infty} \sin(x) = \text{does not exist}$$

since $\sin(x)$ oscillates between -1 and 1 .

Evaluate

$$\lim_{x \rightarrow \infty} \frac{3 \sin(x)}{x^2} = 0$$

since the denominator grows to infinity while $-3 \leq 3 \sin(x) \leq 3$.

Evaluate

$$\lim_{x \rightarrow \infty} \frac{2x^3 + x^2 \cdot \cos(x) + 3e^x + x}{x^5 + 5e^x} = \frac{3}{5}$$

since the exponential functions grow much faster than the rest.
To use limit laws, multiply numerator and denominator by $\frac{1}{e^x}$.

Infinite Limits at Infinity

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

if we can make the values of $f(x)$ arbitrary large by taking x sufficiently large.

Similar for:

$$\lim_{x \rightarrow \infty} f(x) = -\infty \quad \lim_{x \rightarrow -\infty} f(x) = \infty \quad \lim_{x \rightarrow -\infty} f(x) = -\infty$$

$$\lim_{x \rightarrow \infty} x^3 = \infty$$

$$\lim_{x \rightarrow -\infty} x^3 = -\infty$$

$$\lim_{x \rightarrow \infty} a^x = \infty$$

for $a > 1$

$$\lim_{x \rightarrow -\infty} a^x = \infty$$

for $0 < a < 1$

Infinite Limits at Infinity

Evaluate

$$\lim_{x \rightarrow \infty} (x^2 - x)$$

The limit laws do not help since:

$$\lim_{x \rightarrow \infty} (x^2 - x) = \lim_{x \rightarrow \infty} x^2 - \lim_{x \rightarrow \infty} x = \infty - \infty = \text{invalid expression}$$

However, we can write

$$\lim_{x \rightarrow \infty} (x^2 - x) = \lim_{x \rightarrow \infty} x(x - 1) = \infty$$

because both x and $x - 1$ become arbitrarily large.

Infinite Limits at Infinity: Heuristics

All on this slide is heuristics, not laws!

On the last slide we could have reasoned as follows:

$$\lim_{x \rightarrow \infty} (x^2 - x) = \lim_{x \rightarrow \infty} x \cdot \lim_{x \rightarrow \infty} (x - 1) = \infty \cdot \infty = \infty$$

Valid calculations with ∞ and x a real number:

$$\infty + \infty = \infty \qquad \infty + x = \infty \qquad \frac{x}{\infty} = 0$$

$$\frac{\infty}{x} = \infty \text{ if } x > 0 \qquad \frac{\infty}{x} = -\infty \text{ if } x < 0$$

Invalid, undefined expressions:

$$\infty - \infty \qquad \infty + (-\infty) \qquad \frac{\infty}{\infty} \qquad 0 \cdot \infty$$

Infinite Limits at Infinity

Evaluate

$$\lim_{x \rightarrow \infty} \frac{x^2 + x}{3 - x}$$

We have

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{x^2 + x}{3 - x} &= \lim_{x \rightarrow \infty} \left(\frac{x^2 + x}{3 - x} \cdot \frac{1}{\frac{1}{x}} \right) \\ &= \lim_{x \rightarrow \infty} \frac{x + 1}{\frac{3}{x} - 1} \\ &= \frac{\infty}{0 - 1} \\ &= -\infty\end{aligned}$$

because $x + 1$ grows to infinity while $\frac{3}{x} - 1$ gets closer to -1 .