

# MATH 211

## Online Asynchronous Survey in Calculus and Analytical Geometry

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# Derivative as a Function

The **derivative of  $f$**  is a function  $f'$  defined by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

- ▶ The domain of  $f'$  is the set  $\{x \mid f'(x) \text{ exists}\}$ .
- ▶ Geometrically,  $f'(x)$  is the slope of the tangent at  $(x, f(x))$ .

Let  $f(x) = x^3 - x$ . Find a formula for  $f'(x)$ .

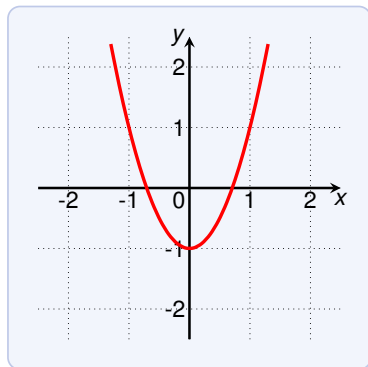
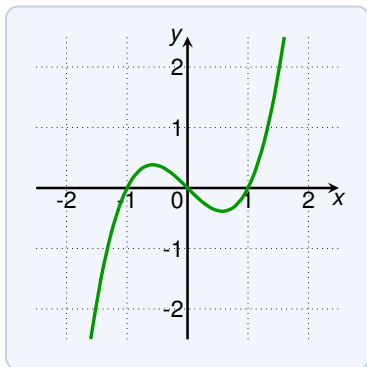
$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} = \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2 - 1) \\ &= 3x^2 - 1 \end{aligned}$$

Using the definition of derivative, find  $f'(x)$ , where  $f(x) = \sqrt{2x}$ .

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{2x+2h} - \sqrt{2x}}{h} \\ &= \lim_{h \rightarrow 0} \left( \frac{\sqrt{2x+2h} - \sqrt{2x}}{h} \cdot \frac{\sqrt{2x+2h} + \sqrt{2x}}{\sqrt{2x+2h} + \sqrt{2x}} \right) \\ &= \lim_{h \rightarrow 0} \left( \frac{2x + 2h - 2x}{h \cdot (\sqrt{2x+2h} + \sqrt{2x})} \right) \\ &= \lim_{h \rightarrow 0} \left( \frac{2}{\sqrt{2x+2h} + \sqrt{2x}} \right) \\ &= \frac{2}{2\sqrt{2x}} = \frac{1}{\sqrt{2x}} \end{aligned}$$

# Derivative as a Function

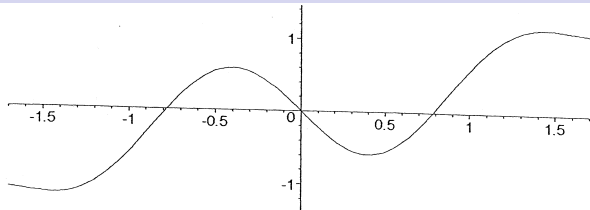
Which of these functions is the derivative of the other?



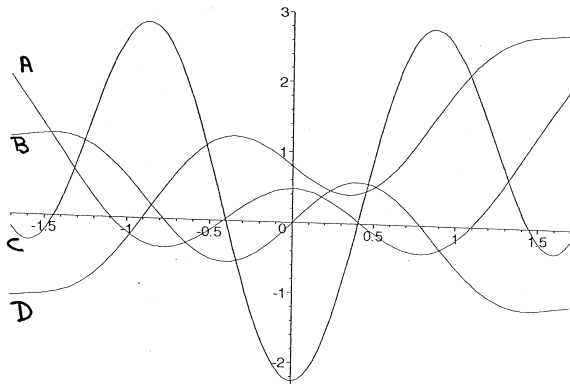
The right is the derivative of the left:

- ▶ look at local maxima and minima of  $f$ ; then  $f'$  must be 0
- ▶ where  $f$  increases,  $f'$  must be positive
- ▶ where  $f$  decreases,  $f'$  must be negative

# Exam Task from 2005



Which of the four labelled curves below best represents the graph of  $y = f'(x)$ ?



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# Derivative as a Function

Practice computing derivatives!

Try examples from the book for yourself. Among others:

- ▶ Example 3 in Section 2.8
- ▶ Example 4 in Section 2.8

# Derivative as a Function

A function  $f$  is **differentiable at**  $a$  if  $f'(a)$  exists.

A function  $f$  is **differentiable on an open interval**  $(a, b)$  if it is differentiable at every number of the interval.

Note that the interval  $(a, b)$  may be  $(-\infty, b)$ ,  $(a, \infty)$  or  $(-\infty, \infty)$ .

# Derivative as a Function

Where is  $f(x) = |x|$  differentiable?

For  $x > 0$  we have:

- ▶  $|x| = x$ ,
- ▶  $|x + h| = x + h$  for small enough  $h$ .

Thus for  $x > 0$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{x+h-x}{h} = \lim_{h \rightarrow 0} 1 = 1$$

For  $x < 0$  we have:

- ▶  $|x| = -x$ ,
- ▶  $|x + h| = -x - h$  for small enough  $h$ .

Thus for  $x < 0$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{-x-h+x}{h} = \lim_{h \rightarrow 0} -1 = -1$$



# Derivative as a Function

Where is  $f(x) = |x|$  differentiable?

For  $x = 0$

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{|h|}{h}$$

We need to look at the left and right limits:

$$\lim_{h \rightarrow 0^-} \frac{|h|}{h} \stackrel{\text{since } h < 0}{=} \lim_{h \rightarrow 0^-} \frac{-h}{h} = \lim_{h \rightarrow 0^-} -1 = -1$$

and

$$\lim_{h \rightarrow 0^+} \frac{|h|}{h} \stackrel{\text{since } h > 0}{=} \lim_{h \rightarrow 0^+} \frac{h}{h} = \lim_{h \rightarrow 0^+} 1 = 1$$

The left and right limits are different.

Thus  $f'(0)$  does not exist, and  $f(x)$  is not differentiable at 0.

Hence  $f$  is differentiable at all numbers in  $(-\infty, 0) \cup (0, \infty)$ .

# Derivatives and Continuity

If  $f$  is differentiable at  $a$ , then  $f$  is continuous at  $a$ .

The proof is in the book. Intuitively it holds because...

Differentiable at  $a$  means:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \quad \text{exists}$$

Continuous at  $a$  means:

$$\begin{aligned} \lim_{x \rightarrow a} f(x) = f(a) &\iff \lim_{x \rightarrow a} (f(x) - f(a)) = 0 \\ &\iff \lim_{h \rightarrow 0} (f(a+h) - f(a)) = 0 \end{aligned}$$

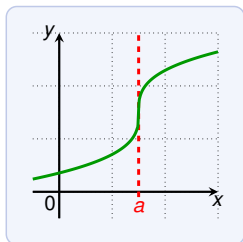
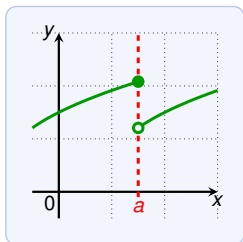
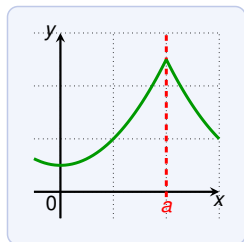
If the latter limit would not be 0 (or not exist), then  $\frac{f(a+h)-f(a)}{h}$  would get arbitrarily large for small  $h$ .

If  $f$  is continuous at  $a$ , then  $f$  is **not always** differentiable at  $a$ .

E.g.  $|x|$  is continuous at 0 but not differentiable at 0.

# How can a Function fail to be Derivable?

There are the following reasons for failure of being derivable:



- ▶ graph changes direction abruptly (graph has a “corner”)
- ▶ the function is not continuous at  $a$
- ▶ graph has a vertical tangent at  $a$ , that is:

$$\lim_{x \rightarrow a} |f'(x)| = \infty$$

Example for a vertical tangent is  $f(x) = \sqrt[3]{x}$  at 0.

## Derivative: Other Notations

We usually write  $f'(x)$  for the derivative.

However, there are other common notations:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The symbols  $\frac{d}{dx}$  and  $D$  are called **differentiation operators**. (they indicate the operation of computing the derivative)

The notation  $\frac{dy}{dx}$  has been introduced by Leibnitz:

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

In Leibnitz notation  $f'(a)$  is written as

$$\left. \frac{dy}{dx} \right|_a \quad \text{or} \quad \left. \frac{dy}{dx} \right]_a$$

# Higher Derivatives

If  $f$  is a function, the derivative  $f'$  is also a function.

Thus we can compute the derivative of the derivative:

$$(f')' = f''$$

The function  $f''$  is called **second derivative** of  $f$ .

Let  $f(x) = x^3 - x$ . Find  $f''(x)$ .

We have seen  $f'(x) = 3x^2 - 1$ . Thus

$$\begin{aligned} f''(x) &= (f')'(x) = \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[3(x+h)^2 - 1] - [3x^2 - 1]}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2 + 6xh + 3h^2 - 1 - 3x^2 + 1}{h} \\ &= \lim_{h \rightarrow 0} \frac{6xh + 3h^2}{h} = \lim_{h \rightarrow 0} (6x + 3h) = 6x \end{aligned}$$

# Higher Derivatives

What is the meaning of  $f''(x)$ ?

- ▶ the slope of  $f'(x)$  at point  $(x, f'(x))$
- ▶ the rate of change of  $f'(x)$
- ▶ the rate of change of the rate of change of  $f(x)$

The **acceleration** is an example of a second derivative:

- ▶  $s(t)$  is the position of an object (at time  $t$ )
- ▶  $v(t) = s'(t)$  is the speed (at time  $t$ )
- ▶  $a(t) = v'(t) = s''(t)$  is the acceleration (at time  $t$ )

# Higher Derivatives

We can continue this process of deriving:

- ▶  $f'''(x) = (f'')'(x)$
- ▶  $f''''(x) = (f''')'(x)$
- ▶ ...

The  $n$ -th derivative of  $f$  is denoted by

$$f^{(n)}(x) \qquad \text{or} \qquad \frac{d^n y}{dx^n}$$

For example,  $f = f^{(0)}$ ,  $f' = f^{(1)}$ ,  $f'' = f^{(2)}$ ,  $f''' = f^{(3)}$

Let  $f(x) = x^3 - x$ . Find  $f'''(x)$  and  $f^{(4)}(x)$ .

We know  $f''(x) = 6x$ . Hence

$$f'''(x) = 6 \qquad f^{(4)}(x) = 0$$

Note that  $f'''$  is the slope of  $f''$ , and  $f^{(4)}$  is the slope of  $f'''$ .