MATH 211
Online Asynchronous Survey in Calculus and Analytical Geometry

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Derivatives of Logarithmic Functions

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

Proof.

Let $y = \log_a x$. Then

$$a^y = x$$

Using implicit differentiation we get:

$$\frac{d}{dx}a^y = \frac{d}{dx}x \quad \Longrightarrow \quad \ln a \cdot a^y \cdot y' = 1$$

$$\Longrightarrow \quad y' = \frac{1}{\ln a \cdot a^y} = \frac{1}{x \ln a}$$

$\square$

From the formula it follows that

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Derivatives of Logarithmic Functions

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

Differentiate

$$y = \ln(x^3 + 1)$$

We have

$$y' = \frac{1}{x^3 + 1} \cdot 3x^2$$

Differentiate

$$y = \ln(\sin x)$$

We have

$$y' = \frac{1}{\sin x} \cdot \cos x = \cot x$$

Derivatives of Logarithmic Functions

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

Differentiate

$$y = \sqrt{\ln x}$$

We have

$$y' = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x}$$

Differentiate

$$y = \log_{10}(2 + \sin x)$$

We have

$$y' = \frac{1}{(2 + \sin x) \ln 10} \cdot \cos x$$

Derivatives of Logarithmic Functions

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

Differentiate

$$f(x) = \ln|x|$$

We have

$$f(x) = \begin{cases} \ln x & \text{for } x > 0 \\ \ln(-x) & \text{for } x < 0 \end{cases}$$

Thus

$$f'(x) = \begin{cases} \frac{1}{x} & \text{for } x > 0 \\ \frac{1}{-x} \cdot (-1) = \frac{1}{x} & \text{for } x < 0 \end{cases}$$

Hence

$$\frac{d}{dx} \ln|x| = \frac{1}{x} \quad \text{for all } x \neq 0$$

Derivatives of Logarithmic Functions

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

Differentiate

$$y = \ln \frac{x+1}{\sqrt{x-2}}$$

We have

$$\begin{aligned}
y' &= \frac{1}{\frac{x+1}{\sqrt{x-2}}} \cdot \frac{d}{dx} \frac{x+1}{\sqrt{x-2}} \\
&= \frac{\sqrt{x-2}}{x+1} \cdot \frac{1 \cdot \sqrt{x-2} - (x+1) \cdot \frac{d}{dx}\sqrt{x-2}}{(\sqrt{x-2})^2} \\
&= \frac{\sqrt{x-2}}{x+1} \cdot \frac{\sqrt{x-2} - (x+1) \cdot \frac{1}{2\sqrt{x-2}} \cdot 1}{(\sqrt{x-2})^2} \\
&= \frac{x-2-(x+1)\cdot\frac{1}{2}}{(x+1)(x-2)} = \frac{x-5}{2(x+1)(x-2)}
\end{aligned}$$

Derivatives of Logarithmic Functions

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

Differentiate

$$y = \ln \frac{x+1}{\sqrt{x-2}}$$

We have

$$\begin{aligned} y &= \ln(x+1) - \ln\sqrt{x-2} \\ &= \ln(x+1) - \frac{1}{2}\ln(x-2) \end{aligned}$$

Thus

$$y' = \frac{1}{x+1} - \frac{1}{2} \cdot \frac{1}{x-2}$$

Logarithmic Differentiation

The following method is called **logarithmic differentiation.**

Differentiate

$$y = \frac{x^{\frac{3}{4}} \cdot \sqrt{x^2+1}}{(3x+2)^5}$$

We take logarithms on both sides:

$$\begin{aligned}\ln y = \ln \frac{x^{\frac{3}{4}} \cdot \sqrt{x^2+1}}{(3x+2)^5} &= \ln x^{\frac{3}{4}} + \ln \sqrt{x^2+1} - \ln(3x+2)^5 \\ &= \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)\end{aligned}$$

We use implicit differentiation:

$$\frac{d}{dx}\ln y = \frac{3}{4}\frac{d}{dx}\ln x + \frac{1}{2}\frac{d}{dx}\ln(x^2+1) - 5\frac{d}{dx}\ln(3x+2)$$

$$\frac{1}{y}y' = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot 2x - 5\frac{1}{3x+2}\cdot 3$$

Logarithmic Differentiation

The following method is called **logarithmic differentiation.**

Differentiate

$$y = \frac{x^{\frac{3}{4}} \cdot \sqrt{x^2+1}}{(3x+2)^5}$$

We have:

$$\frac{1}{y}y' = \frac{3}{4} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{1}{x^2+1} \cdot 2x - 5\frac{1}{3x+2} \cdot 3$$

Solving for y' yields:

$$y' = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

Hence

$$y' = \frac{x^{\frac{3}{4}} \cdot \sqrt{x^2+1}}{(3x+2)^5} \cdot \left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

Logarithmic Differentiation

Steps of **Logarithmic Differentiation**:

- Take natural logarithms on both sides of $y = f(x)$.
- Use laws of logarithms to simplify.
- Differentiate implicitly with respect to x.
- Solve the resulting equation for y'.

Logarithmic Differentiation

Differentiate

$$y = x^{\sqrt{x}}$$

The power rule does not apply: the exponent contains x!

We use logarithmic differentiation:

$$\ln y = \ln\left(x^{\sqrt{x}}\right) = \sqrt{x} \cdot \ln x$$

$$\frac{d}{dx} \ln y = \frac{d}{dx} \left(\sqrt{x} \cdot \ln x\right)$$

$$\frac{1}{y} y' = \sqrt{x} \cdot \frac{1}{x} + \ln x \cdot \frac{1}{2\sqrt{x}}$$

$$y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = y\left(\frac{2 + \ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right)$$

Alternative: $x^{\sqrt{x}} = (e^{\ln x})^{\sqrt{x}} = e^{\ln x \cdot \sqrt{x}}$ now use chain rule

The Number e as a Limit

Let $f(x) = \ln x$. We know that

$$f'(x) = \frac{1}{x} \qquad \text{and hence} \qquad f'(1) = 1$$

By definition of the limit

$$\begin{aligned} 1 = f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\ln(1+h) - \ln(1)}{h} \\ &= \lim_{h \to 0} \left(\frac{1}{h} \cdot \ln(1+h) \right) = \lim_{h \to 0} \ln(1+h)^{\frac{1}{h}} \end{aligned}$$

As a consequence we get

$$e = e^1 = e^{f'(1)} = e^{\lim_{h \to 0} \ln(1+h)^{\frac{1}{h}}} = \lim_{h \to 0} e^{\ln(1+h)^{\frac{1}{h}}} = \lim_{h \to 0} (1+h)^{\frac{1}{h}}$$

$$e \quad = \quad \lim_{h \to 0} (1+h)^{\frac{1}{h}} \quad = \quad \lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n$$