MATH 211
Online Asynchronous Survey in Calculus and Analytical Geometry

Dr. Ahmed Kaffel

Department of Mathematical Sciences
University of Wisconsin Milwaukee

Spring 2023

UNIVERSITY of WISCONSIN
UWMILWAUKEE

Related (Dependent) Rates

Air is pumped into a spherical balloon:

- the volume increases with 100 cm^3/s

Find: rate of change of the radius when the diameter is $50cm$.

First step: introduce suggestive notation

- let $V(t)$ be the volume after time t
- let $r(t)$ be the radius after time t

Then the given problem translates to

$$V'(t) = 100 \text{ cm}^3/\text{s} \qquad \text{Find } r'(t) \text{ when } r = 25\text{cm}.$$

How are the volume of a sphere and its radius related?

$$V = \frac{4}{3}\pi r^3 \quad \text{thus} \quad V'(t) = \frac{d}{dt}\left(\frac{4}{3}\pi r(t)^3\right) = \frac{4}{3}\pi \cdot 3r(t)^2 r'(t)$$

We solve for $r'(t)$:

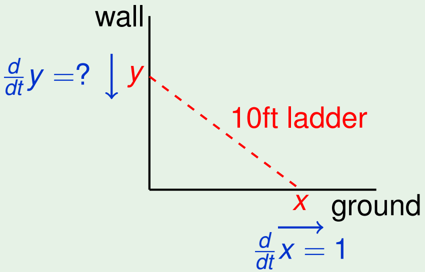
$$r'(t) = \frac{V'(t)}{4\pi \cdot r(t)^2} \qquad r'(t) = \frac{100}{4\pi \cdot 25^2} = \frac{1}{25\pi} \text{ cm/s}$$

Related (Dependent) Rates

A ladder of length 10ft rests against a vertical wall.

- the bottom of the ladder slides away from the wall with 1ft/s

How fast is the top sliding when the bottom is 6ft from the wall?



Thus

$$x^2 + y^2 = 10^2$$
$$\overset{x=6}{\Longrightarrow} 6^2 + y^2 = 10^2$$
$$\Longrightarrow y = \pm\sqrt{10^2 - 6^2}$$
$$\Longrightarrow y = 8$$

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}10^2 \implies 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$
$$\implies \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} \implies \frac{dy}{dt} = -\frac{6}{8} \cdot 1 = -\frac{3}{4}$$

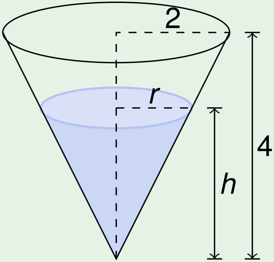
The top slides with $\frac{3}{4}$ft/s when the bottom is 6ft from the wall.

Related (Dependent) Rates

A water tank has the shape of an inverted circular cone:

- base radius $2m$ and the height is $4m$,
- water is pumped into the tank at a rate of $2\text{m}^3/\text{min}$.

At what rate is the water rising when the water is 3m deep?



$$V = \frac{1}{3}\pi r^2 h$$

How is r related to h?

$$\frac{r}{h} = \frac{2}{4} \quad \Longrightarrow \quad r = \frac{1}{2}h$$

$$V = \frac{1}{3}\pi(\frac{1}{2}h)^2 h = \frac{1}{12}\pi h^3$$

We differentiate both sides with respect to t:

$$\frac{dV}{dt} = \frac{d}{dt}(\frac{1}{12}\pi h^3) = \frac{1}{12}\pi 3h^2\frac{dh}{dt} \quad \Longrightarrow \quad \frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt} \overset{h=3}{=} \frac{4}{\pi 9} \cdot 2$$

Thus the water rises with $8/(\pi 9)$m/min when its is 3m deep.

Related (Dependent) Rates

Problem Solving Strategy

Important when solving textual problems:

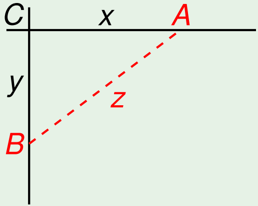
- Read the problem carefully.
- Draw a diagram.
- Introduce notation, function names for the quantities.
- Express given information and goal using the notation.
- Write equations relating the quantities. Eliminate dependent variables (in the previous example we have eliminated the radius as it was dependent on the height).
- Use the chain rule to differentiate both sides w.r.t. t.
- Solve for the unknown rate, and substitute the given information into the resulting formula.

Related (Dependent) Rates

Two cars are headed for the same road intersection:

- car A is traveling west with 50mi/h
- car B is traveling north with 60mi/h

At what rate are the cars approaching
when A is 0.3mi and B is 0.4mi from the intersection?



- $x(t) =$ distance of A to crossing
- $y(t) =$ distance of B to crossing
- $z(t) =$ distance of A to B

$$\frac{d}{dt}x = -50 \qquad \frac{d}{dt}y = -60$$

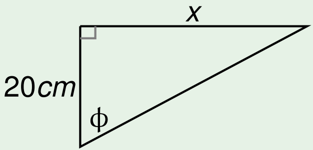
$$z^2 = x^2 + y^2 \implies 2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt} + \frac{y}{z}\frac{dy}{dt} \implies \frac{dz}{dt} = \frac{0.3}{0.5}(-50) + \frac{0.4}{0.5}(-60) = -78$$

When $x = 0.3$ & $y = 0.4$, we get $z = 0.5$. The answer is 78mi/h.

Related (Dependent) Rates

We have a right-angled triangle of the form



The length x increases with $4 cm/s$.
How fast is the angle ϕ changing when $x = 15cm$?

The quantities x and ϕ are related by:

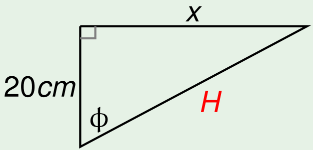
$$\tan\phi = \frac{x}{20}$$

Differentiating both sides yields:

$$\frac{d}{dt}\tan\phi = \frac{d}{dt}\frac{x}{20} \implies \frac{1}{(\cos\phi)^2}\cdot\frac{d\phi}{dt} = \frac{1}{20}\cdot\frac{dx}{dt}$$

Related (Dependent) Rates

We have a right-angled triangle of the form



The length x increases with $4cm/s$.
How fast is the angle ϕ changing when $x = 15cm$?

$$\frac{1}{(\cos\phi)^2} \cdot \frac{d\phi}{dt} = \frac{1}{20} \cdot \frac{dx}{dt}$$

$$\implies \quad \frac{d\phi}{dt} = \frac{(\cos\phi)^2}{20} \cdot \frac{dx}{dt} = \frac{(\cos\phi)^2}{20} \cdot 4 = \frac{(\cos\phi)^2}{5}$$

We have $\cos\phi = 20/H = 20/\sqrt{15^2 + 20^2} = 20/25 = 4/5$.
Thus

$$\frac{d\phi}{dt} = \left(\frac{4}{5}\right)^2 \cdot \frac{1}{5} = \frac{4^2}{5^3} = \frac{16}{125}\text{rad/s}$$