MATH 211
Online Asynchronous Survey in Calculus and Analytical Geometry

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Review problems

Find the derivative of $f(x)$ when $0 < x < 5$:

$$f(x) = \frac{|x|}{\sqrt{5^2 - x^2}}$$

If $0 < x$ then $|x| = x$. Then

$$\begin{aligned} f(x) &= \frac{x}{\sqrt{5^2 - x^2}} \\ f'(x) &= \frac{1 \cdot \sqrt{5^2 - x^2} - x \cdot \frac{1}{2}(5^2 - x^2)^{-\frac{1}{2}} \cdot (-2x)}{(\sqrt{5^2 - x^2})^2} \\ &= \frac{(5^2 - x^2) + x^2}{(\sqrt{5^2 - x^2})^3} = \frac{25}{(5^2 - x^2)^{\frac{3}{2}}} \end{aligned}$$

What is the left-hand derivative at 0? Then $x < 0$, thus $|x| = -x$.

$$\text{left-hand derivative at } 0 = -1/5$$

Review problems

Find the derivative of

$$f(x) = \left(\frac{1}{x^3} - 3\frac{1}{x}\right) \cdot (2x^4 + 5x)$$

using the product rule.

$$f'(x) = \left(\frac{1}{x^3} - 3\frac{1}{x}\right) \cdot \frac{d}{dx}(2x^4 + 5x) + (2x^4 + 5x)\frac{d}{dx}\left(\frac{1}{x^3} - 3\frac{1}{x}\right)$$

$$= \left(\frac{1}{x^3} - 3\frac{1}{x}\right) \cdot (8x^3 + 5) + (2x^4 + 5x)\left(-3x^{-4} - 3 \cdot (-1)x^{-2}\right)$$

$$= \left(\frac{1}{x^3} - 3\frac{1}{x}\right) \cdot (8x^3 + 5) + (2x^4 + 5x)\left(\frac{-3}{x^4} + \frac{3}{x^2}\right)$$

$$= \left(8 - 24x^2 + \frac{5}{x^3} - \frac{15}{x}\right) + \left(-6 + 6x^2 - \frac{15}{x^3} + \frac{15}{x}\right)$$

$$= 2 - 18x^2 - \frac{10}{x^3}$$

Review problems

Find the derivative of

$$f(x) = \frac{\sqrt{x} - 2}{\sqrt{x} + 2}$$

using the quotient rule.

$$\begin{aligned} f'(x) &= \frac{(\sqrt{x} + 2)\frac{d}{dx}(\sqrt{x} - 2) - (\sqrt{x} - 2)\frac{d}{dx}(\sqrt{x} + 2)}{(\sqrt{x} + 2)^2} \\ &= \frac{(\sqrt{x} + 2)\frac{1}{2\sqrt{x}} - (\sqrt{x} - 2)\frac{1}{2\sqrt{x}}}{(\sqrt{x} + 2)^2} \\ &= \frac{(\sqrt{x} + 2) - (\sqrt{x} - 2)}{2\sqrt{x}(\sqrt{x} + 2)^2} \\ &= \frac{4}{2\sqrt{x}(\sqrt{x} + 2)^2} = \frac{2}{\sqrt{x}(\sqrt{x} + 2)^2} \end{aligned}$$

Review problems

Find the derivative of

$$f(x) = \frac{\sqrt{x^3} + 4}{3x^2 + 7}$$

using the quotient rule.

$$f'(x) = \frac{(3x^2 + 7) \cdot \frac{d}{dx}(\sqrt{x^3} + 4) - (\sqrt{x^3} + 4) \cdot \frac{d}{dx}(3x^2 + 7)}{(3x^2 + 7)^2}$$

$$= \frac{(3x^2 + 7) \cdot \frac{d}{dx}(x^{\frac{3}{2}} + 4) - (\sqrt{x^3} + 4) \cdot \frac{d}{dx}(3x^2 + 7)}{(3x^2 + 7)^2}$$

$$= \frac{(3x^2 + 7) \cdot \frac{3}{2} \cdot x^{\frac{1}{2}} - (\sqrt{x^3} + 4) \cdot 6x}{(3x^2 + 7)^2}$$

Find the derivative of

$$f(x) = 2x \cdot \sin(x) \cdot \cos(x)$$

We have:

$$\begin{aligned}
f(x) &= 2x \cdot \big(\sin(x) \cdot \cos(x)\big) \\
f'(x) &= 2x \cdot \frac{d}{dx}\big(\sin(x) \cdot \cos(x)\big) + \big(\sin(x) \cdot \cos(x)\big) \cdot \frac{d}{dx}2x \\
&= 2x \cdot \big(\sin(x) \cdot (-\sin(x)) + \cos(x) \cdot \cos(x)\big) \\
&\qquad + \big(\sin(x) \cdot \cos(x)\big) \cdot 2 \\
&= 2x \cdot \big(\cos^2(x) - \sin^2(x)\big) + 2\sin(x)\cos(x) \\
&= 2x \cdot \cos(2x) + \sin(2x)
\end{aligned}$$

Find the derivative of

$$f(x) = 8x^2 + 2e^x$$

We have:

$$f'(x) = 16x + 2e^x$$

Find the derivative of

$$f(x) = x \cdot e^{\frac{1}{x}}$$

We have:

$$\begin{aligned} f'(x) &= x \cdot \frac{d}{dx} e^{\frac{1}{x}} + e^{\frac{1}{x}} \cdot \frac{d}{dx} x \\ &= x \cdot e^{\frac{1}{x}} \cdot \frac{d}{dx} \frac{1}{x} + e^{\frac{1}{x}} \\ &= x \cdot e^{\frac{1}{x}} \cdot (-1) x^{-2} + e^{\frac{1}{x}} \\ &= e^{\frac{1}{x}} (1 - \frac{1}{x}) \end{aligned}$$

Find the derivative of

$$f(x) = \frac{8e^x}{e^x + 1}$$

We have:

$$\begin{aligned} f'(x) &= \frac{(e^x + 1) \cdot \frac{d}{dx}(8e^x) - 8e^x \cdot \frac{d}{dx}(e^x + 1)}{(e^x + 1)^2} \\ &= \frac{(e^x + 1) \cdot 8e^x - 8e^x \cdot e^x}{(e^x + 1)^2} \\ &= \frac{8e^x}{(e^x + 1)^2} \end{aligned}$$

Find the derivative of

$$f(x) = \cos(\sin(x^2))$$

We have:

$$\begin{aligned} f'(x) &= -\sin(\ \sin(x^2)\ ) \cdot \frac{d}{dx}(\sin(x^2)) \\ &= -\sin(\ \sin(x^2)\ ) \cdot \cos(x^2) \cdot \frac{d}{dx}x^2 \\ &= -\sin(\ \sin(x^2)\ ) \cdot \cos(x^2) \cdot 2x \end{aligned}$$

Use logarithmic differentiation to find the derivative of

$$y = \sqrt{\frac{x-1}{x^4+1}}$$

We have:

$$\ln y = \ln \sqrt{\frac{x-1}{x^4+1}} = \frac{1}{2} \cdot \ln \frac{x-1}{x^4+1} = \frac{1}{2} \cdot \Big(\ln(x-1) - \ln(x^4+1) \Big)$$

Thus

$$\frac{d}{dx} \ln y = \frac{d}{dx} \left[\frac{1}{2} \cdot \Big(\ln(x-1) - \ln(x^4+1) \Big) \right]$$

$$\frac{1}{y} y' = \frac{1}{2} \cdot \left(\frac{d}{dx} \ln(x-1) - \frac{d}{dx} \ln(x^4+1) \right)$$

$$y' = \frac{1}{2} y \cdot \left(\frac{1}{x-1} - \frac{1}{x^4+1} 4x^3 \right)$$

$$y' = \frac{1}{2} \cdot \sqrt{\frac{x-1}{x^4+1}} \cdot \left(\frac{1}{x-1} - \frac{1}{x^4+1} 4x^3 \right)$$

Find the derivative of

$$y = \sqrt{x} \cdot (1 + x^2)^{\sin(x)}$$

We have:

$$\ln y = \ln\left[\sqrt{x} \cdot (1 + x^2)^{\sin(x)}\right] = \ln\sqrt{x} + \ln(1 + x^2)^{\sin(x)}$$

$$= \frac{1}{2} \cdot \ln x + \sin(x) \cdot \ln(1 + x^2)$$

Thus

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left[\frac{1}{2} \cdot \ln x + \sin(x) \cdot \ln(1 + x^2)\right]$$

$$\frac{y'}{y} = \left[\frac{1}{2x} + \sin(x) \cdot \frac{d}{dx}\ln(1 + x^2) + \ln(1 + x^2) \cdot \frac{d}{dx}\sin(x)\right]$$

$$y' = y\left[\frac{1}{2x} + \sin(x)\frac{1}{1 + x^2}2x + (\ln(1 + x^2)) \cdot \cos(x)\right]$$

$$y' = \left(\sqrt{x} \cdot (1 + x^2)^{\sin(x)}\right)\left[\frac{1}{2x} + \frac{2x\sin(x)}{1 + x^2} + \cos(x)\ln(1 + x^2)\right]$$

Let $f(x) = x^2$. Find $a > 0$ such that the tangent to the curve at the point $(a, f(a))$ passes through the point $(1, -3)$.

We have $f'(x) = 2x$. Thus the tangent through $(a, f(a))$ is

$$y - f(a) = f'(a) \cdot (x - a) \quad \Longrightarrow \quad y - a^2 = 2a(x - a)$$

We want that the tangent passes through $(1, -3)$, thus

$$\begin{aligned} & (-3) - a^2 = 2a(1 - a) \\ \Longrightarrow \quad & (-3) - a^2 = 2a - 2a^2 \\ \Longrightarrow \quad & a^2 - 2a - 3 = 0 \\ \Longrightarrow \quad & a = 1 \pm \sqrt{1 + 3} \end{aligned}$$

Thus $a = 3$ (recall that we were searching for $a > 0$)
The tangent is $y - 9 = 6(x - 3)$

Show that $f(x) = 2e^x + 3x + 15x^3$ has no tangent with slope 2.

We have:

$$f'(x) = 2e^x + 3 + 5x^2$$

Note that

$$e^x \geq 0 \quad \text{for all } x$$
$$x^2 \geq 0 \quad \text{for all } x$$

and thus

$$f'(x) = 2e^x + 3 + 15x^2 \geq 3$$

The slope of the curve $f(x)$ is ≥ 3 everywhere.
Hence the curve cannot have a tangent with slope 2.

Where does the normal line to $f(x) = x - x^2$ at point $(1, 0)$ intersect the the curve the second time?

We have:

$$f'(x) = 1 - 2x \quad \Longrightarrow \quad f'(1) = -1$$

The normal line at $(1, 0)$ has slope $-\frac{1}{-1} = 1$, and thus is

$$y - 0 = 1 \cdot (x - 1) \qquad\qquad y = x - 1$$

We look for the intersection of $f(x)$ and the normal line:

$$\begin{aligned} x - 1 = x - x^2 \quad &\Longrightarrow \quad x^2 - 1 = 0 \\ &\Longrightarrow \quad (x - 1) \cdot (x + 1) = 0 \end{aligned}$$

Thus the second intersection is at point $(-1, -2)$.

Find constants A, B, C such that $y = Ax^2 + Bx + C$ satisfies

$$y'' + y' - 2y = x^2$$

We have

$$y = Ax^2 + Bx + C \qquad y' = 2Ax + B \qquad y'' = 2A$$

Thus

$$\begin{aligned} x^2 = y'' + y' - 2y &= (2Ax + B) + (2A) - 2(Ax^2 + Bx + C) \\ &= (-2A)x^2 + (2A - 2B)x + (2A + B - 2C) \end{aligned}$$

Hence

$$\begin{aligned} -2A = 1 &\implies A = -1/2 \\ 2A - 2B = 0 &\implies B = A = -1/2 \\ 2A + B - 2C = 0 &\implies C = -3/4 \end{aligned}$$

Let $c > \frac{1}{2}$. How many lines through the point $(0, c)$ are normal lines to $f(x) = x^2$?

Let a be arbitrary. We construct the normal line at $(a, f(a))$:

$$f'(a) = 2a$$

As a consequence the normal line at $(a, f(a))$ for $a \neq 0$ is:

$$y - a^2 = -\frac{1}{2a}(x - a)$$

We check for which a the normal goes through $(0, c)$:

$$c - a^2 = -\frac{1}{2a}(0 - a) \quad \Longrightarrow \quad a^2 = c - \frac{1}{2}$$

Note that $c - \frac{1}{2} > 0$! Thus there are two solutions for a.
Note that the normal at $(0, 0)$ is vertical and goes through $(0, c)$!
Hence there are three normal lines going through $(0, c)$.

Is there a line that is tangent to both curves f and g?

$$f(x) = x^2 \qquad\qquad g(x) = x^2 - 2x + 2$$

We compute the tangent to f at $(a, f(a))$:

$$f'(a) = 2a \quad \Longrightarrow \quad y - a^2 = 2a(x - a)$$

$$\Longrightarrow \quad y = (2a)x + (-a^2)$$

We compute the tangent to g at $(b, g(b))$:

$$g'(b) = 2b - 2 \quad \Longrightarrow \quad y - (b^2 - 2b + 2) = (2b - 2)(x - b)$$

$$\Longrightarrow \quad y = (2b - 2)x + (-b^2 + 2)$$

The tangents are equal if:

$$2a = 2b - 2 \quad \Longrightarrow \quad a = b - 1 \quad \Longrightarrow \quad a = 1/2$$

$$-a^2 = -b^2 + 2 \quad \Longrightarrow \quad -(b - 1)^2 = -b^2 + 2$$

$$\Longrightarrow \quad -(b^2 - 2b + 1) = -b^2 + 2 \quad \Longrightarrow \quad b = 3/2$$

Thus the line $y = x - 1/4$ is tangent to both curves.

Find all points on the curve where the slope of the tangent is -1

$$x^2y^2 + xy = 2$$

We use implicit derivatives:

$$\frac{d}{dx}(x^2y^2 + xy) = \frac{d}{dx}2$$

$$\implies \quad x^2\frac{d}{dx}y^2 + y^2\frac{d}{dx}x^2 \quad + \quad x\frac{d}{dx}y + y\frac{d}{dx}x = 0$$

$$\implies \quad x^2 2yy' + y^2 2x \quad + \quad xy' + y = 0$$

$$\overset{y'=-1}{\implies} \quad -2x^2y + 2xy^2 - x + y = 0$$

$$\implies \quad (-x + y) \cdot (2xy + 1) = 0$$

Find all points on the curve where the slope of the tangent is -1

$$x^2y^2 + xy = 2$$

The slope is -1 if $x = y$ or $xy = -1/2$.

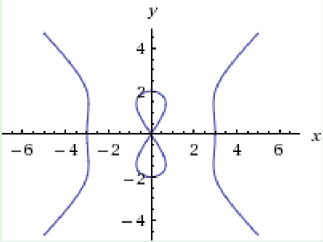
But when is (x, y) on the original curve?

$$\begin{aligned}
x = y &\implies (x^2)^2 + x^2 = 2 \\
&\implies (x^2 - 1)(x^2 + 2) = 0 \\
&\implies x^2 = 1 \\
&\implies x = \pm 1 \\
&\implies \text{on the curve if } x = y = \pm 1 \\
xy = -1/2 &\implies (xy)^2 + xy2 \\
&\implies 1/4 + 1/2 = 2 \\
&\implies \text{can never be on the curve}
\end{aligned}$$

The points on the curve with slope -1 are $(1, 1)$ and $(-1, -1)$.

Find an equation for the tangent at point $(3, -2)$ to the curve

$$y^2(y^2 - 4) = x^2(x^2 - 9)$$



We have

$$y^4 - 4y^2 = x^4 - 9x^2$$

We use implicit differentiation

$$\frac{d}{dx}(y^4 - 4y^2) = \frac{d}{dx}(x^4 - 9x^2)$$

$$\implies \quad 4y^3y' - 8yy' = 4x^3 - 18x$$

$$\implies \quad y'(4y^3 - 8y) = 4x^3 - 18x$$

$$\implies \quad y' = \frac{2x^3 - 9x}{2y^3 - 4y} = \frac{2 \cdot 3^3 - 9 \cdot 3}{2(-2)^3 - 4 \cdot (-2)} = \frac{54 - 27}{-16 + 8} = -\frac{27}{8}$$

Thus the equation for the tangent is $y + 2 = -\frac{27}{8} \cdot (x - 3)$.

Evaluate the limit

$$\lim_{x \to \infty} (x \cdot \sin(\frac{1}{x}))$$

We have

$$\lim_{x \to \infty} (x \cdot \sin(\frac{1}{x})) \overset{\text{take } h = \frac{1}{x}}{=} \lim_{h \to 0} (\frac{1}{h} \cdot \sin(h)) = 1$$

Find the derivative of

$$\begin{aligned} f(x) &= (\sqrt{x^5} - 3\sqrt[3]{x}) \cdot (6x^4 + 2x) \\ &= (x^{\frac{5}{2}} - 3x^{\frac{1}{3}}) \cdot (6x^4 + 2x) \end{aligned}$$

We have:

$$\begin{aligned} f'(x) &= (x^{\frac{5}{2}} - 3x^{\frac{1}{3}}) \cdot \frac{d}{dx}(6x^4 + 2x) + (6x^4 + 2x) \cdot \frac{d}{dx}(x^{\frac{5}{2}} - 3x^{\frac{1}{3}}) \\ &= (x^{\frac{5}{2}} - 3x^{\frac{1}{3}}) \cdot (24x^3 + 2) + (6x^4 + 2x) \cdot (\frac{5}{2}x^{\frac{3}{2}} - x^{-\frac{2}{3}}) \end{aligned}$$