

MATH 211

Online Asynchronous Survey in Calculus and Analytical Geometry

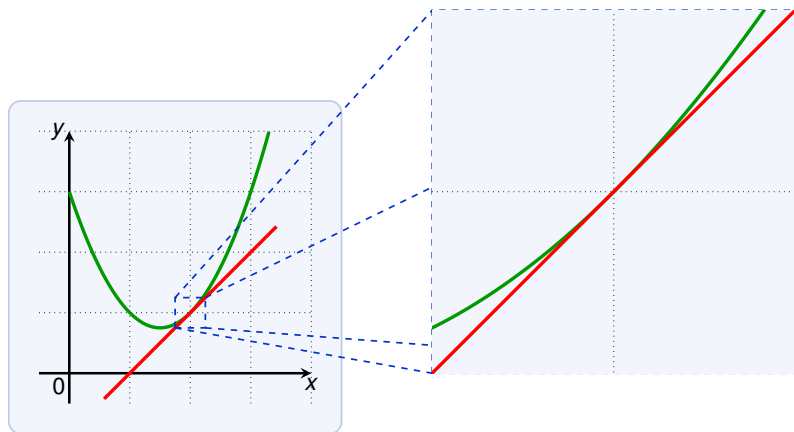
Dr. Ahmed Kaffel

Department of Mathematical Sciences
University of Wisconsin Milwaukee

Spring 2023



Linear Approximation and Differentials



A curve is very close to its tangent close to the point of tangency (touching).

We can use this for approximating values of the function...

Linear Approximation and Differentials

Why approximate values of a function using a tangent?

- ▶ might be easy to compute $f(a)$ and $f'(a)$,
- ▶ but difficult to compute values $f(x)$ with x near a

We use the tangent line at $(a, f(a))$ to approximate $f(x)$ when x is close to a .

The tangent at $(a, f(a))$ is:

$$L(x) = f(a) + f'(a) \cdot (x - a)$$

This function is called **linearization** of f at a .

When x is close to a , we approximate $f(x)$ by:

$$f(x) \approx f(a) + f'(a) \cdot (x - a)$$

This is called

- ▶ **linear approximation** of f at a , or
- ▶ **tangent line approximation** of f at a .

Linear Approximation and Differentials

Find the linearization of $f(x) = \sqrt{x+3}$ at 1 and use it to approximate $\sqrt{3.98}$.

We have:

$$f(1) = \sqrt{3+1} = 2$$

$$f'(x) = \frac{1}{2\sqrt{x+3}} \qquad f'(1) = \frac{1}{2\sqrt{1+3}} = \frac{1}{4}$$

Thus the linearization of f at 1 is:

$$L(x) = 2 + \frac{1}{4}(x-1)$$

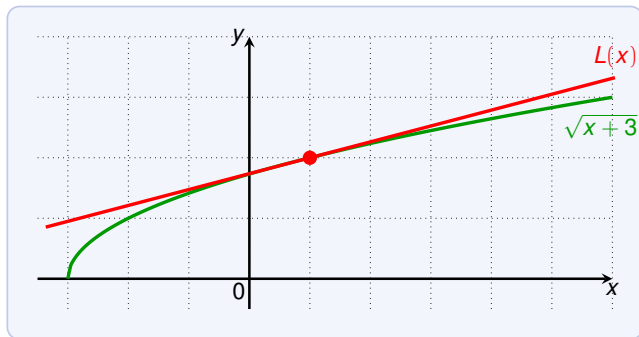
Thus for x close to 1 we approximate $f(x)$ by:

$$f(x) = \sqrt{x+3} \approx 2 + \frac{1}{4}(x-1)$$

In particular:

$$\sqrt{3.98} = \sqrt{0.98+3} \approx 2 + \frac{1}{4}(0.98-1) = 2 - 0.005 = 1.995$$

Linear Approximation and Differentials



The linear approximation is close to the curve when x is near 1.

Linear Approximation and Differentials

What is the linear approximation of $f(x) = \sin x$ at 0?
Use it to approximate $\sin 0.01$.

We have:

$$f(0) = \sin 0 = 0$$

$$f'(x) = \cos x$$

$$f'(0) = 1$$

Thus the linear approximation of $\sin x$ at 0 is:

$$L(x) = 0 + 1(x - 0) = x$$

We use this to approximate $\sin 0.01$:

$$\sin 0.01 \approx L(0.01) = 0.01$$

Linear Approximation and Differentials

What is the linear approximation of $f(x) = \cos x$ at 0?
Use it to approximate $\cos 0.01$.

We have:

$$\begin{aligned}f(0) &= \cos 0 = 1 \\f'(x) &= -\sin x & f'(0) &= 0\end{aligned}$$

Thus the linear approximation of $\cos x$ at 0 is:

$$L(x) = 1 + 0(x - 0) = 1$$

We use this to approximate $\cos 0.01$:

$$\cos 0.01 \approx L(0.01) = 1$$

Approximations for \sin and \cos are often applied in physics (e.g. optics).

Linear Approximation and Differentials

Final Exam 2005

Use differential approximation, or the linearization method, to approximate $\sqrt[4]{15.5}$.

We have $f(x) = \sqrt[4]{x}$.

We need to choose where to compute the linearization: $a = 16$.

$$f(16) = 2$$

$$f'(x) = \frac{1}{4}x^{-\frac{3}{4}} \quad f'(16) = \frac{1}{4}16^{-\frac{3}{4}} = \frac{1}{4}\sqrt[4]{16}^{-3} = \frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$$

The linearization of f at 16 is:

$$L(x) = 2 + \frac{1}{32}(x - 16)$$

Then the approximation of $\sqrt[4]{15.5}$ is:

$$\sqrt[4]{15.5} \approx L(15.5) = 2 + \frac{1}{32}(15.5 - 16) = 2 - \frac{1}{64} = \frac{127}{64}$$

Linear Approximation and Differentials

Final Exam 2004

Use the linearization method to approximate $(1.98)^4$.

We have $f(x) = x^4$.

We need to choose where to compute the linearization: $a = 2$.

$$f(2) = 16$$

$$f'(x) = 4x^3 \qquad f'(2) = 4 \cdot 2^3 = 4 \cdot 8 = 32$$

The linearization of f at 2 is:

$$L(x) = 16 + 32(x - 2)$$

Then the approximation of $(1.98)^4$ is:

$$\begin{aligned}(1.98)^4 &\approx L(1.98) = 16 + 32(1.98 - 2) = 16 + 32(-0.02) \\ &= 16 + 32\left(-\frac{1}{50}\right) = 16 - \frac{16}{25} = \frac{16 \cdot 24}{25}\end{aligned}$$

Linear Approximation and Differentials

Final Exam 2003 (Spring)

Use differentials or the linearization approximation method to approximate $\ln(0.9)$.

We have $f(x) = \ln x$.

We need to choose where to compute the linearization: $a = 1$.

$$f(1) = 0$$

$$f'(x) = \frac{1}{x}$$

$$f'(1) = 1$$

The linearization of f at 1 is:

$$L(x) = 0 + 1(x - 1) = x - 1$$

Then the approximation of $\ln(0.9)$ is:

$$\ln(0.9) \approx L(0.9) = 0.9 - 1 = -0.1$$

Linear Approximation and Differentials

Final Exam 2003 (Fall)

Use differentials to approximate $\sqrt[3]{999}$.

We have $f(x) = \sqrt[3]{x}$.

We choose where to compute the linearization: $a = 1000$.

$$f(1000) = 10$$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3(\sqrt[3]{x})^2} \quad f'(1000) = \frac{1}{3 \cdot 10^2} = \frac{1}{300}$$

The linearization of f at 1000 is:

$$L(x) = 10 + \frac{1}{300}(x - 1000)$$

Then the approximation of $\sqrt[3]{999}$ is:

$$\sqrt[3]{999} \approx L(999) = 10 + \frac{1}{300}(999 - 1000) = 10 - \frac{1}{300} = \frac{2999}{300}$$

Linear Approximation and Differentials

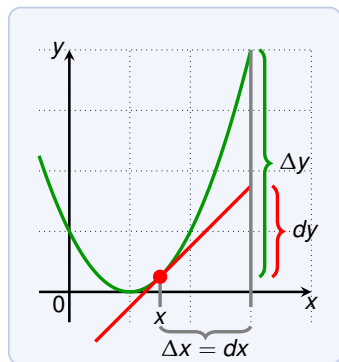
The method of linear approximation with differentials:

$$f'(x) = \frac{dy}{dx}$$

We view dx and dy as variables, then:

$$dy = f'(x) dx$$

So dy depends on the value of x and dx .



- ▶ x = point of linearization
- ▶ $\Delta x = dx$ is the distance from x
- ▶ dy = change of y of tangent
- ▶ Δy = change of y of curve f

As formulas:

- ▶ $dy = f'(x) dx$
- ▶ $\Delta y = f(x + \Delta x) - f(x)$