# MATH 211
# Online Asynchronous Survey in Calculus and Analytical Geometry

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Spring 2023

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# Maximum and Minimum Values

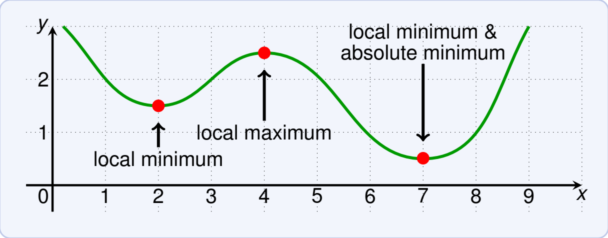
An important application of derivatives are

**optimization problems**,

that is, finding the best way of doing something.

These problems can often be reduces to finding the minimum or maximum of a function.

# Maximum and Minimum Values




Let $c$ be in the domain $D$ of $f$. Then $f(c)$ is the

- **absolute maximum** value of $f$ if $f(c) \geq f(x)$ for all $x$ in $D$
- **absolute minimum** value of $f$ if $f(c) \leq f(x)$ for all $x$ in $D$

Often called **global maximum** or **global minimum.**
Minima and maxima are called **extreme values** of $f$.

The number $f(c)$ is a

- **local maximum** value of $f$ if $f(c) \geq f(x)$ when $x$ is near $c$
- **local minimum** value of $f$ if $f(c) \leq f(x)$ when $x$ is near $c$

# Maximum and Minimum Values

Where does

$$f(x) = x^2$$

have local / global minima or maxima?

The value $f(0) = 0$ is absolute and local minimum since:

$$f(0) = 0 \leq x^2 = f(x) \quad \text{for all } x$$

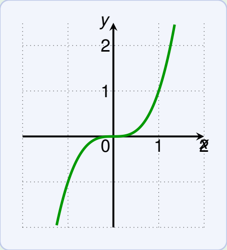
The function has no local or global maxima.

Where does

$$f(x) = x^3$$

have (local or global) minima or maxima?

The function has no local or global extrema.

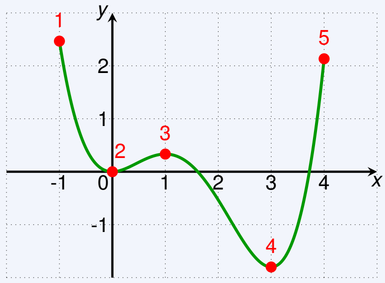


# Maximum and Minimum Values

The graph of

$$f(x) = \frac{3x^4 - 16x^3 + 18x^2}{15}$$

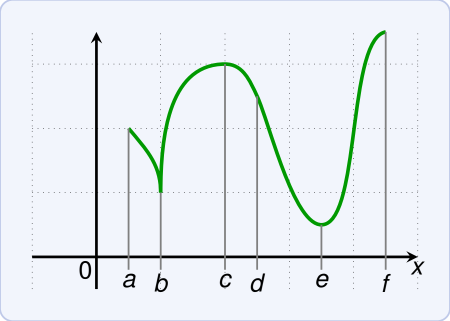
for $-1 \leq x \leq 4$ is shown in this diagram:




Which of the points are a local / global maxima or minima?

1. global (absolute) maximum;
   not a local maximum since $f$ is not defined near $-1$
2. local minimum
3. local maximum
4. global (absolute) and local minimum
5. nothing

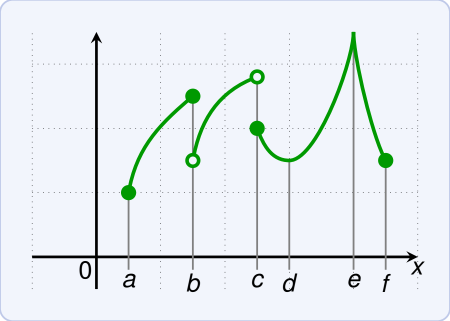
# Maximum and Minimum Values




Which of the points are global/local maxima/minima?

- *a* nothing
- *b* local minimum
- *c* local maximum
- *d* nothing
- *e* local and global (absolute) minimum
- *f* global (absolute) maximum, but not a local maximum

# Maximum and Minimum Values




Which of the points are global/local maxima/minima?

- *a* global (absolute) minimum, but not a local minimum
- *b* local maximum
- *c* nothing
- *d* local minimum
- *e* local and global (absolute) maximum
- *f* nothing

# Maximum and Minimum Values

Let $f$ be a function, and $[a, b]$ a closed interval. Then $f(c)$ is an

- **absolute maximum** on $[a, b]$ if $f(c) \geq f(x)$ for all $x$ in $[a, b]$
- **absolute minimum** on $[a, b]$ if $f(c) \leq f(x)$ for all $x$ in $[a, b]$

## Extreme Value Theorem

If $f$ is continuous on a closed interval $[a, b]$, then

- $f$ has an absolute maximum $f(c)$ for some $c$ in $[a, b]$,
- $f$ has an absolute minimum $f(d)$ for some $d$ in $[a, b]$.


Continuous on $[1, 7]$.

Absolute minimum:
$f(4) = 1$

Absolute maximum:
$f(2) = 3$, and
$f(6) = 3$

# Maximum and Minimum Values

## Extreme Value Theorem

If $f$ is continuous on a closed interval $[a, b]$, then

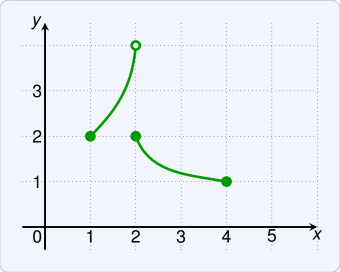
- $f$ has an absolute maximum $f(c)$ for some $c$ in $[a, b]$,
- $f$ has an absolute minimum $f(d)$ for some $d$ in $[a, b]$.


Continuous on $[1, 6]$.

Absolute minimum:
$f(6) = 1$

Absolute maximum:
$f(3) = 3$

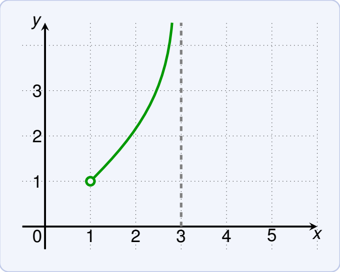
# Maximum and Minimum Values




Absolute minimum:
$f(4) = 1$

Absolute maximum:
none

Not continuous on $[1, 4]$!




Absolute minimum:
none

Absolute maximum:
none

Continuous on $(1, 3)$, but this is not a closed interval!

The function needs to be **continuous** on a **closed** interval $[a, b]$.

# Maximum and Minimum Values

## Fermat's Theorem

If $f$ has a local maximum or minimum at $c$ and $f'(c)$ exists, then $f'(c) = 0$.

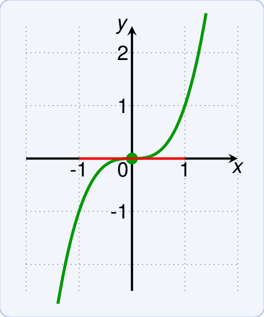

At every **local** maximum or minimum, the tangent is horizontal. (if the derivative exists)

# Maximum and Minimum Values

## Fermat's Theorem

If $f$ has a local maximum or minimum at $c$ and $f'(c)$ exists, then $f'(c) = 0$.

The reverse statement is not true! Having $f'(c) = 0$ does not guarantee that $f(c)$ is a minimum or maximum.




For example:

$$f(x) = x^3$$

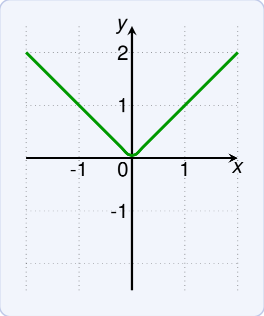
Then $f'(0) = 0$.
But there is no minimum or maximum.

# Maximum and Minimum Values

### Fermat's Theorem

If $f$ has a local maximum or minimum at $c$ and $f'(c)$ exists, then $f'(c) = 0$.

A local minimum/maximum does not guarantee that $f'(c)$ exists.




For example:

$$f(x) = |x|$$

Then $f(0) = 0$ is a local minimum.
But $f'(0)$ does not exist.

Care needed for applying the theorem (check both conditions)!

# Maximum and Minimum Values

## Fermat's Theorem

If $f$ has a local maximum or minimum at $c$ and $f'(c)$ exists, then $f'(c) = 0$.

The theorem suggests where local extra can occur:

- where $f'(c) = 0$, or
- where $f'(c)$ does not exist.

A **critical number** of a function $f$ is a number $c$ in the domain of $f$ such that either $f'(c) = 0$, or $f'(c)$ does not exist.

What are the critical numbers of $f(x) = x^{3/5}(5 - x)$?

$$f(x) = x^{3/5}(5 - x) = 5x^{3/5} - x^{8/5}$$

$$f'(x) = \frac{3}{x^{2/5}} - \frac{8}{5}x^{3/5} = \frac{15}{5x^{2/5}} - \frac{8x}{5x^{2/5}} = \frac{15 - 8x}{5x^{2/5}}$$

The critical numbers are $\frac{15}{8}$ $(f(c) = 0)$ and 0 $(f(c)$ does not exist)

# Maximum and Minimum Values

## Fermat's Theorem

If $f$ has a local maximum or minimum at $c$ and $f'(c)$ exists, then $f'(c) = 0$.

What are the critical numbers of the function

$$f(x) = \sqrt{x} + |x - 2| \qquad ?$$

Due to $|x - 2|$, the derivative is not defined at x = 2 .

For $x < 2$ we have $|x - 2| = -(x - 2)$, thus:

$$f(x) = \sqrt{x} - (x - 2) \qquad\qquad f'(x) = \frac{1}{2\sqrt{x}} - 1$$

Thus $f'(x) = 0 \iff x = 1/4$, and $f'(x)$ undefined for $x = 0$.

For $x > 2$ we have $|x - 2| = x - 2$, thus:

$$f(x) = \sqrt{x} + (x - 2) \qquad\qquad f'(x) = \frac{1}{2\sqrt{x}} + 1 \geq 1$$

Thus the critical numbers are $0$, $1/4$ and $2$.

# Maximum and Minimum Values

## Fermat's Theorem

If $f$ has a local maximum or minimum at $c$ and $f'(c)$ exists, then $f'(c) = 0$.

We can now rephrase the the theorem as follows:

If $f$ has a local extremum at $c$, then $c$ is a critical number of $f$.

We can use this to look for global extrema on intervals:

## Closed Interval Method

To find the **absolute** maximum and minimum values of a continuous function $f$ on an closed interval $[a, b]$:

1. Find the values of $f$ at critical numbers of $f$ in $(a, b)$.
2. Find the values of $f$ at the endpoints of the interval.
3. The largest value of (1) and (2) is the absolute maximum, the lowest the absolute minimum.

# Maximum and Minimum Values

Find the absolute absolute maximum and minimum values of

$$f(x) = x^3 - 3x^2 + 1 \qquad -\frac{1}{2} \le x \le 4$$

Since $f$ is cont. on $[-\frac{1}{2}, 4]$ we can use Closed Interval Method.

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

We have $f'(x) = 0$ if $x = 0$ or $x = 2$. Both in $[-\frac{1}{2}, 4]$!
No other critical values since $f'(x)$ exists for all $x$.

The values of $f$ at the critical numbers are:

$$f(0) = 1 \qquad f(2) = -3$$

The values of $f$ at the end points of the interval are:

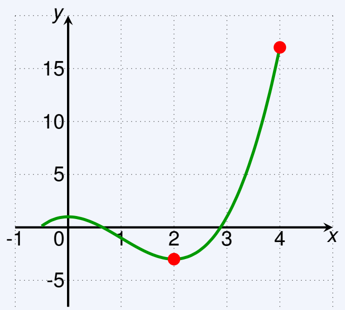
$$f(-\frac{1}{2}) = -\frac{1}{8} - 3\frac{1}{4} + 1 = \frac{1}{8} \qquad f(4) = 4 \cdot 16 - 3 \cdot 16 + 1 = 17$$

Absolute minimum is $f(2) = -3$, absolute maximum $f(4) = 17$.

# Maximum and Minimum Values

Find the absolute absolute maximum and minimum values of

$$f(x) = x^3 - 3x^2 + 1 \qquad -\frac{1}{2} \leq x \leq 4$$




Absolute minimum is $f(2) = -3$, absolute maximum $f(4) = 17$.

# Maximum and Minimum Values

Assume that an object is moving with speed

$$v(t) = (t-1)^3 - 4t^2 + 9t + 5 \qquad\qquad 0 \le t \le 5$$

Find the absolute minimum and maximum acceleration.

The acceleration is:

$$a(t) = v'(t) = 3(t-1)^2 - 8t + 9 = 3t^2 - 14t + 12$$

Since $a$ is cont. on $[0,5]$ we can use Closed Interval Method.

$$a'(t) = 6t - 14 \qquad\qquad a'(t) = 0 \iff t = \frac{7}{3}$$

The only critical number is $\frac{7}{3}$. Note that $\frac{7}{3}$ is in $[0,5]$.
No other critical numbers since $a'(t)$ is defined everywhere.

# Maximum and Minimum Values

Assume that an object is moving with speed

$$v(t) = (t-1)^3 - 4t^2 + 9t + 5 \qquad\qquad 0 \le t \le 5$$

Find the absolute minimum and maximum acceleration.

The acceleration is:

$$a(t) = v'(t) = 3t^2 - 14t + 12$$
$$a'(t) = 6t - 14 \qquad\qquad a'(t) = 0 \iff t = \frac{7}{3}$$

The values at critical numbers and end points of the interval:

$$a\left(\frac{7}{3}\right) = 3\left(\frac{7}{3}\right)^2 - 14\frac{7}{3} + 12 = \frac{7 \cdot 7}{3} - \frac{14 \cdot 7}{3} + \frac{36}{3} = -\frac{13}{3}$$
$$a(0) = 12$$
$$a(5) = 3 \cdot 5^2 - 14 \cdot 5 + 12 = 15 \cdot 5 - 14 \cdot 5 + 12 = 17$$

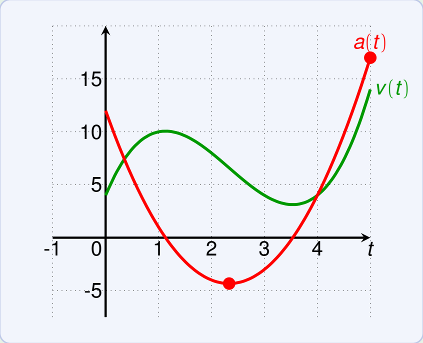
The absolute minimum acceleration is $a\left(\frac{7}{3}\right) = -\frac{13}{3}$.
The absolute maximum acceleration is $a(5) = 17$.

# Maximum and Minimum Values

Assume that an object is moving with speed

$$v(t) = (t-1)^3 - 4t^2 + 9t + 5 \qquad 0 \le t \le 5$$

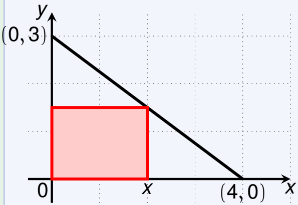
Find the absolute minimum and maximum acceleration.




The absolute minimum acceleration is $a\left(\frac{7}{3}\right) = -\frac{13}{3}$.
The absolute maximum acceleration is $a(5) = 17$.

# Practice problem

Find the area of the largest rectangle that can be inscribed as shown in the triangle.




The line trough $(0,3)$ & $(4,0)$ has the equation: $\ell(x) = -\frac{3}{4}x + 3$

The area $A$ of the rectangle depends on the width $x$:

$$A(x) = x \cdot \ell(x) = x \cdot (-\frac{3}{4}x + 3) = -\frac{3}{4}x^2 + 3x \quad \text{for } x \text{ in } [0,4]$$

$$A'(x) = -\frac{3}{2}x + 3 \qquad A'(x) = 0 \iff \frac{3}{2}x = 3 \iff x = 2$$

Thus the only critical number is 2. The value of $A(x)$ at 0, 2, 4:

$$A(0) = 0 \qquad A(2) = 3 \qquad A(4) = 0$$

The the area of the largest rectangle is 3.