MATH 211
Online Asynchronous Survey in Calculus and Analytical Geometry

Dr. Ahmed Kaffel

Department of Mathematical Sciences
University of Wisconsin Milwaukee

Spring 2023

UNIVERSITY of WISCONSIN UWMILWAUKEE

Derivatives and the Shape of a Graph

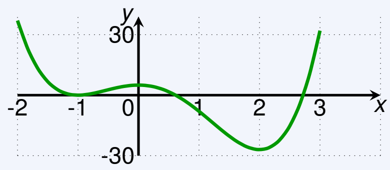
If $f'(x) > 0$ on an interval, then f is increasing on that interval.

If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Where is $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ increasing/decreasing?

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1)$$

Interval	$12x$	$x-2$	$x+1$	$f'(x)$	
x < -1	-	-	-	-	decreasing on $(-\infty, -1)$
-1 < x < 0	-	-	+	+	increasing on $(-1, 0)$
0 < x < 2	+	-	+	-	decreasing on $(0, 2)$
2 < x	+	+	+	+	increasing on $(2, \infty)$



Recall Fermat's Theorem

If f has a local extremum at c, then c is a critical number.

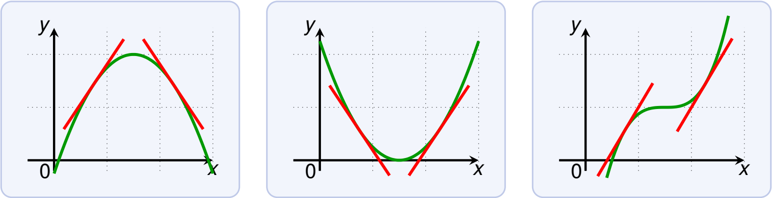
But not ever critical number is an extremum. We need a test!

Derivatives and the Shape of a Graph

First Derivative Test

Suppose that c is a critical number of a continuous function f.

- If f' changes the sign from positive to negative, then f has a local maximum at c.
- If f' changes the sign from negative to positive, then f has a local minimum at c.
- If f' does not change sign at c, then f has no local extremum at c.



Derivatives and the Shape of a Graph

What are the local extrema of $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$?

$$f'(x) = 12x(x-2)(x+1)$$

The critical numbers are: -1, 0 and 2.

We have already seen that:

Interval	$12x$	$x-2$	$x+1$	$f'(x)$	
x < -1	-	-	-	-	decreasing on $(-\infty, -1)$
-1 < x < 0	-	-	+	+	increasing on $(-1, 0)$
0 < x < 2	+	-	+	-	decreasing on $(0, 2)$
2 < x	+	+	+	+	increasing on $(2, \infty)$

We have:

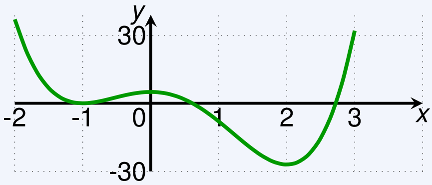
- $f(-1) = 0$ is a local minimum (f' changes from $-$ to $+$)
- $f(0) = 5$ is a local maximum (f' changes from $+$ to $-$)
- $f(2) = -27$ is a local minimum (f' changes from $-$ to $+$)

Derivatives and the Shape of a Graph

What are the local extrema of $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$?

$$f'(x) = 12x(x-2)(x+1)$$

The critical numbers are: -1, 0 and 2.



We have:

- $f(-1) = 0$ is a local minimum (f' changes from $-$ to $+$)
- $f(0) = 5$ is a local maximum (f' changes from $+$ to $-$)
- $f(2) = -27$ is a local minimum (f' changes from $-$ to $+$)

Derivatives and the Shape of a Graph

What are the local extrema of

$$f(x) = x + 2\sin x \qquad\qquad 0 \le x \le 2\pi \quad ?$$

We have

$$f'(x) = 1 + 2\cos x$$

$$f'(x) = 0 \iff \cos x = -\frac{1}{2} \iff x = \frac{2\pi}{3} \text{ or } x = \frac{4\pi}{3}$$

As f' is defined everywhere these are the only critical numbers.

Interval	$f'(x)$	
$0 < x < \frac{2\pi}{3}$	+	increasing on $(0, \frac{2\pi}{3})$
$\frac{2\pi}{3} < x < \frac{4\pi}{3}$	-	decreasing on $(\frac{2\pi}{3}, \frac{4\pi}{3})$
$\frac{4\pi}{3} < x < 2\pi$	+	increasing on $(\frac{4\pi}{3}, 2\pi)$

As a consequence:

- $f(\frac{2\pi}{3}) = \frac{2\pi}{3} + \sqrt{3}$ is a local maximum (f' from $+$ to $-$)
- $f(\frac{4\pi}{3}) = \frac{4\pi}{3} - \sqrt{3}$ is a local minimum (f' from $-$ to $+$)

Derivatives and the Shape of a Graph

What are the local extrema of

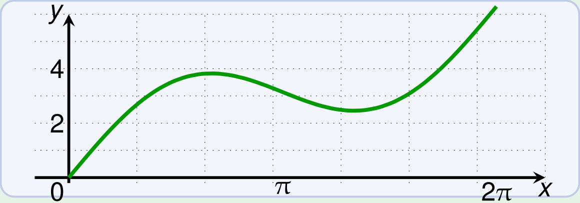
$$f(x) = x + 2\sin x \qquad\qquad 0 \le x \le 2\pi \quad ?$$

We have

$$f'(x) = 1 + 2\cos x$$

$$f'(x) = 0 \iff \cos x = -\frac{1}{2} \iff x = \frac{2\pi}{3} \text{ or } x = \frac{4\pi}{3}$$

As f' is defined everywhere these are the only critical numbers.



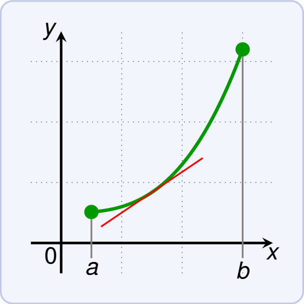
As a consequence:

- $f(\frac{2\pi}{3}) = \frac{2\pi}{3} + \sqrt{3}$ is a local maximum (f' from $+$ to $-$)
- $f(\frac{4\pi}{3}) = \frac{4\pi}{3} - \sqrt{3}$ is a local minimum (f' from $-$ to $+$)

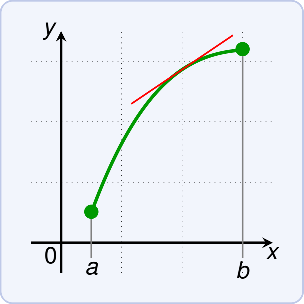
Derivatives and the Shape of a Graph

Let I be an interval. If the graph of f is called

- **concave up** on I if it it lies above all its tangents on I
- **concave down** on I if it it lies below all its tangents on I



concave up

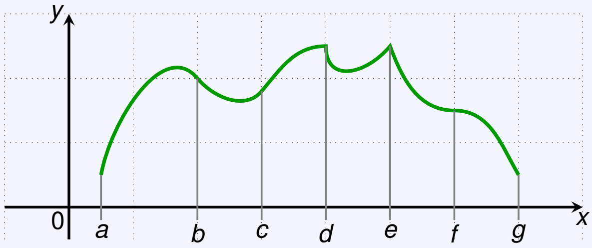


concave down

Imagine the graph as a street & a car driving from left to right:

- then concave upward = turning left (increasing slope)
- then concave downward = turning right (decreasing slope)

Derivatives and the Shape of a Graph



On which interval is the curve concave up / concave down?

- on (a,b) concave downward
- on (b,c) concave upward
- on (c,d) concave downward
- on (d,e) concave upward
- on (e,f) concave upward
- on (f,g) concave downward

Derivatives and the Shape of a Graph

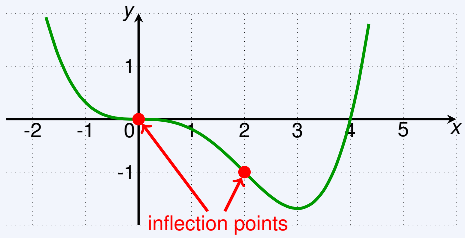
Concavity Test

If $f''(x) > 0$ for all x in I, then f is concave upward on I.

If $f''(x) < 0$ for all x in I, then f is concave downward on I.

A point P on a curve $f(x)$ is called **inflection point** if f is continuous at this point and the curve

- changes from concave upward to downward at P, or
- changes from concave downward to upward at P.



Derivatives and the Shape of a Graph

Where are inflection points of $f(x) = x^4 - 4x^3$?

$$f'(x) = 4x^3 - 12x^2$$

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

Thus $f''(x) = 0$ for $x = 0$ and $x = 2$.

Interval	$f''(x)$	
$x < 0$	+	concave upward on $(-\infty, 0)$
$0 < x < 2$	-	concave downward on $(0, 2)$
$2 < x$	+	concave upward on $(2, \infty)$

Thus the inflection points are:

- $(0, 0)$ since the curve changes from concave up to down
- $(2, -16)$ since the curve changes from concave down to up

Derivatives and the Shape of a Graph

Second Derivative Test

Suppose f'' is continuous near c.

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Where does $f(x) = x^4 - 4x^3$ have local extrema?

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

Thus $f'(x) = 0$ for $x = 0$ and $x = 3$. Second Derivative Test:

$$f''(0) = 0 \qquad\qquad f''(3) = 36 > 0$$

Thus $f(3) = -27$ is a local minimum as $f'(3) = 0$ and $f''(3) > 0$.

The Second Derivative Test gives no information for $f''(0) = 0$.

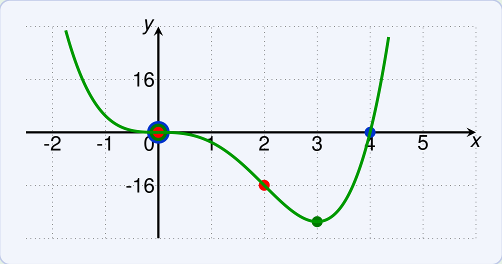
However, the First Derivative Test ... yields that $f(0) = 0$ is no extremum since $f'(x) < 0$ for $x < 0$ and $0 < x < 3$.

Derivatives and the Shape of a Graph

Curve Sketching

$$f(x) = x^4 - 4x^3 = x^3(x-4) \qquad f'(x) = 4x^2(x-3)$$

- $f(x) = 0 \iff x = 0$ or $x = 4$
- local minimum at $(3, -27)$ and $f'(0) = 0$
- inflection points $(0, 0)$ and $(2, -16)$
- decreasing on $(-\infty, 0)$ and $(0, 3)$, increasing on $(3, \infty)$
- concave up on $(-\infty, 0)$, down on $(0, 2)$, up on $(2, \infty)$



Derivatives and the Shape of a Graph

Summary: Finding Local Extrema

Find critical numbers c: $f'(c) = 0$ or $f'(c)$ does not exist.

First **Derivative Test** (f needs to be continuous at c):

- If f' changes from $+$ to $-$ at $c \implies$ local maximum
- If f' changes from $-$ to $+$ at $c \implies$ local minimum
- If f' does not change sign at $c \implies$ no local extremum

The **Second Derivative Test**:

1. $f'(c) = 0$ and $f''(c) > 0 \implies$ local minimum
2. $f'(c) = 0$ and $f''(c) < 0 \implies$ local maximum
3. $f'(c)$ or $f''(c)$ does not exist or $f''(c) = 0$
   $\implies$ use the First Derivative Test