MATH 211
Online Asynchronous Survey in Calculus and Analytical Geometry

Dr. Ahmed Kaffel

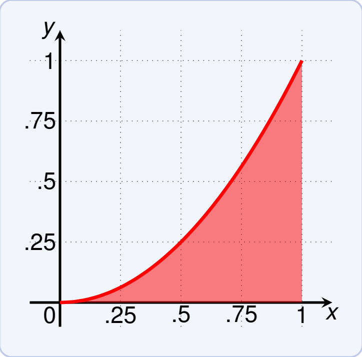
Department of Mathematical Sciences
University of Wisconsin Milwaukee

Spring 2023

UNIVERSITY of WISCONSIN
UWMILWAUKEE

The Area below a Curve

How to compute the area below a curve?

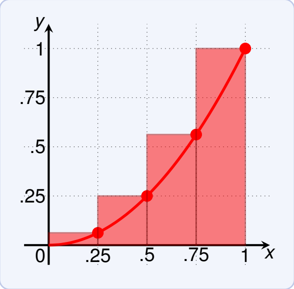


Idea:

- divide the area in vertical strips of equal width
- approximate the area using rectangles

The Area below a Curve

Estimate the area below the curve $f(x) = x^2$ from 0 to 1.



Let's split the area in 4 vertical strips:

- height of each rectangle = value at right endpoint

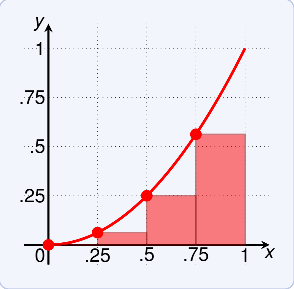
The sum of the area of these rectangles is:

$$R_4 = \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{2}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 + \frac{1}{4} \cdot 1^2 = 0.46875$$

The area A below the curve is less then R_4, that is, $A < R_4$.

The Area below a Curve

Estimate the area below the curve $f(x) = x^2$ from 0 to 1.



Let's split the area in 4 vertical strips:

- height of each rectangle = value at left endpoint

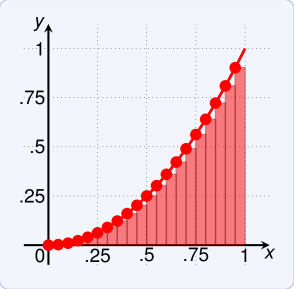
The sum of the area of these rectangles is:

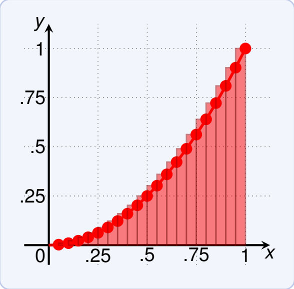
$$L_4 = \frac{1}{4} \cdot 0^2 + \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{2}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 = 0.21875$$

The area A below the curve is larger then L_4, that is, $L_4 < A$.

The Area below a Curve

Estimate the area below the curve $f(x) = x^2$ from 0 to 1.





$$0.21875 = L_4 < A < R_4 = 0.46875$$
$$0.2734375 = L_8 < A < R_8 = 0.3984375$$
$$0.3087500 = L_{20} < A < R_{20} = 0.3587500$$

We have obtained an estimation of A:

- we can improve the estimation by taking more strips

The Area below a Curve

Estimate the area below the curve $f(x) = x^2$ from 0 to 1.

We now let the number of strips go to infinity: $\lim_{n\to\infty} R_n$

The formula for the area R_n with n strips is:

$$\begin{aligned} R_n &= \frac{1}{n}\cdot\left(\frac{1}{n}\right)^2 + \frac{1}{n}\cdot\left(\frac{2}{n}\right)^2 + \frac{1}{n}\cdot\left(\frac{3}{n}\right)^2 + \ldots + \frac{1}{n}\cdot\left(\frac{n}{n}\right)^2 \\ &= \frac{1}{n}\cdot\frac{1}{n^2}(1^2 + 2^2 + 3^2 + \ldots + n^2) \end{aligned}$$

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$R_n = \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

Hence, we have

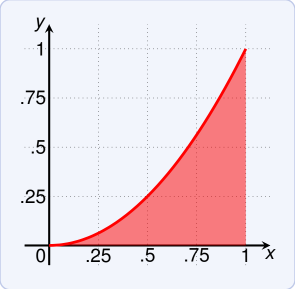
$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) = \frac{2}{6} = \frac{1}{3}$$

The Area below a Curve

Estimate the area below the curve $f(x) = x^2$ from 0 to 1.

$$\lim_{n\to\infty} R_n = \frac{1}{3} \qquad \text{and similar} \qquad \lim_{n\to\infty} L_n = \frac{1}{3}$$

The right- and left-approximations converge to the same value.



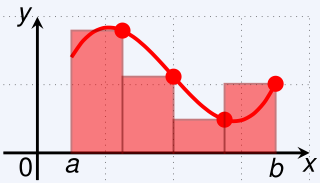
We define the area A to be the limit of these approximations

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \frac{1}{3}$$

The Area below a Curve

Now let's look at a general curve above the x-axis:

The area below the curve of a function f on an interval $[a, b]$.



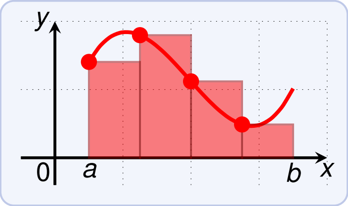
We use n rectangles:

- the width of the interval is $b - a$
- the width of each strip is $\Delta x = (b - a)/n$
- the interval for the i-th strip is $I_i = [a + (i - 1)\Delta x,\ a + i\Delta x]$

The Area below a Curve

Now let's look at a general curve above the x-axis:

The area below the curve of a function f on an interval $[a, b]$.



We use n rectangles: $\Delta x = (b - a)/n$

The area of the rectangles oriented at right-endpoints is:

$$
\begin{aligned}
R_n &= \Delta x \cdot f(a + 1\Delta x) + \Delta x \cdot f(a + 2\Delta x) + \ldots + \Delta x \cdot f(a + n\Delta x) \\
&= \Delta x \big(f(a + 1\Delta x) + f(a + 2\Delta x) + \ldots + f(a + n\Delta x) \big)
\end{aligned}
$$

The area of the rectangles oriented at left-endpoints is:

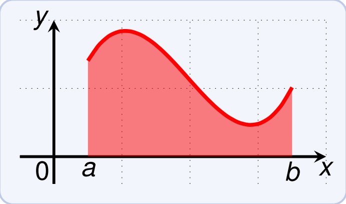
$$
\begin{aligned}
L_n &= \Delta x \cdot f(a + 0\Delta x) + \Delta x \cdot f(a + 1\Delta x) + \ldots + \Delta x \cdot f(a + (n-1)\Delta x) \\
&= \Delta x \big(f(a + 0\Delta x) + f(a + 2\Delta x) + \ldots + f(a + (n-1)\Delta x) \big)
\end{aligned}
$$

The Area below a Curve

The **area** A under the graph of a continuous function f whose graph lies above the x-axis is the limit:

$$A = \lim_{n\to\infty} R_n$$
$$= \lim_{n\to\infty} \left[\Delta x \big(f(a + 1\Delta x) + f(a + 2\Delta x) + \ldots + f(a + n\Delta x)\big)\right]$$

where $\Delta x = (b - a)/n$.



For continuous f this limit always exists, and is the same as

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} \left[\Delta x \big(f(a + 0\Delta x) + \ldots + f(a + (n - 1)\Delta x)\big)\right]$$

The Area below a Curve

Recall that the interval of the i-th strip is:

$$I_i = [a + (i-1)\Delta x,\ a + i\Delta x]$$

$$R_n = \Delta x\big(f(x_1) + f(x_2) + \ldots + f(x_n)\big)$$

where x_i is the right endpoint of the interval I_i

$$L_n = \Delta x\big(f(x_1) + f(x_2) + \ldots + f(x_n)\big)$$

where x_i is the left endpoint of the interval I_i

For continuous curve f above the x-axis we have:

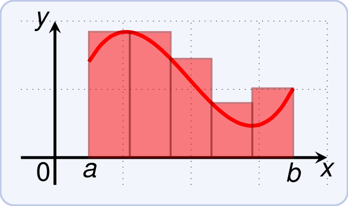
$$A = \lim_{n\to\infty}\left[\Delta x\big(f(x_1) + f(x_2) + \ldots + f(x_n)\big)\right]$$

independent of what **sample points** x_i we take from I_i.

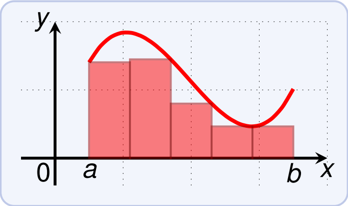
The limit is the same no matter what x_i we choose from I_i!

A famous choice of sample points are upper and lower sums…

The Area below a Curve



upper sum U_5



lower sum D_5

The **upper sum** is

$$U_n = \Delta x \big(f(x_1) + f(x_2) + \ldots + f(x_n) \big)$$

where x_i is chosen from I_i such that $f(x_i)$ is the maximum on I_i

The **lower sum** is

$$D_n = \Delta x \big(f(x_1) + f(x_2) + \ldots + f(x_n) \big)$$

where x_i is chosen from I_i such that $f(x_i)$ is the minimum on I_i

Sigma Notation

We can use the **sigma notation** to write sums more compactly:

$$\sum_{i=1}^{n} h(i) \quad = \quad h(1) + h(2) + \ldots + h(n)$$

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

The Area below a Curve

The area under a curve f above the x-axis from a to b is:

$$A = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \Delta x \cdot f(x_i) \right)$$

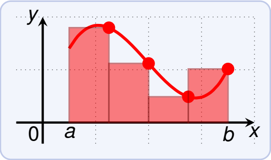
where:

- $\Delta x = (b - a)/n$ is the width of the strips,
- $I_i = [a + (i-1)\Delta x,\ a + i\Delta x]$ is the interval of the i-th strip,
- x_i is the sample point from the i-th interval I_i.

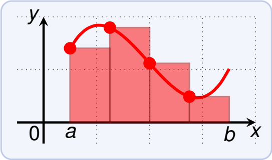
Usual choices for x_i are

- left endpoint $x_i = a + (i-1)\Delta x$ of the interval
- right endpoint $x_i = a + i\Delta x$ of the interval
- middle $x_i = a + (i - \frac{1}{2})\Delta x$ of the interval
- upper sum: $f(x_i)$ is the maximum on the interval I_i
- lower sum: $f(x_i)$ is the minimum on the interval I_i

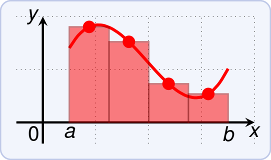
The Area below a Curve



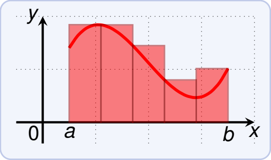
right-endpoints with 5 strips



left-endpoints with 5 strips



midpoints with 5 strips



upper sum with 5 strips

lower sum with 5 strips