MATH 211
Online Asynchronous Survey in Calculus and Analytical Geometry

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The Definite Integral

The **definite integral of f from a to b** is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

provided that the limit exists, and has the same value for all possible choices of the **sample points**

$$x_i \text{ from the interval } [a + (i-1)\Delta x,\ a + i\Delta x]$$

where $\Delta x = \frac{b-a}{n}$.

If the limit exists, we call f **integrable** on $[a, b]$.

The procedure of calculating an integral is called **integration.**

Here a is the **lower limit** and b is the **upper limit** of integration.

The sum $\sum_{i=1}^{n} f(x_i)\Delta x$ is called **Riemann sum.**

The Definite Integral

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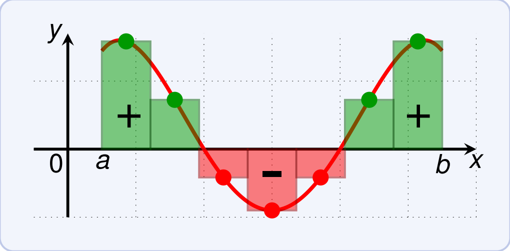


The **Riemann sum** is the sum of the area of rectangles above the x-axis (the green ones) minus the sum of the area of the rectangles below the x-axis (the red ones).

The sample points x_i can be arbitrary from the i-th interval:

- left endpoints, right endpoints or middle of the interval, or
- at maximum (upper sum), or at minimum (lower sum).

The Definite Integral

Evaluate the Riemann sum for

$$f(x) = 2x - 5$$

from 0 to 6 using 3 strips and right endpoints as sample points.

We have:

- the width of the strips is $\Delta x = (6 - 0)/3 = 2$
- the intervals of the strips are $[0, 2]$, $[2, 4]$, $[4, 6]$
- the right endpoints are $x_1 = 2$, $x_2 = 4$, $x_3 = 6$
- the values at x_i's are $f(x_1) = -1$, $f(x_2) = 3$, $f(x_3) = 7$

Thus the Riemann sum using 3 strips and right endpoints is:

$$R_3 = \sum_{i=1}^{3} f(x_i) \cdot \Delta x = 2 \cdot (-1) + 2 \cdot 3 + 2 \cdot 7 = 18$$

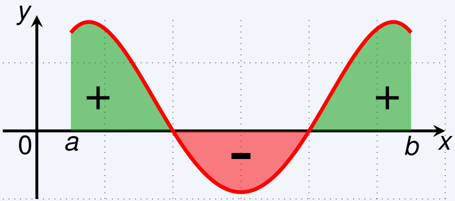
The Definite Integral

The definite integral can be interpreted as the **net area**, that is:

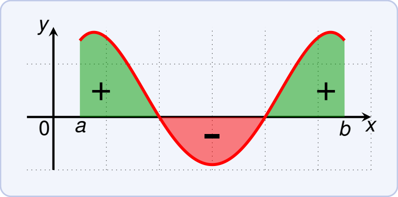
$$\int_a^b f(x)dx = A_1 - A_2$$

where

- A_1 is the area of above the x-axis, below the curve,
- A_2 is the area of below the x-axis, above the curve.



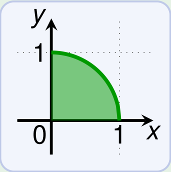
The Definite Integral



Evaluate the integral

$$\int_0^1 \sqrt{1-x^2}dx$$

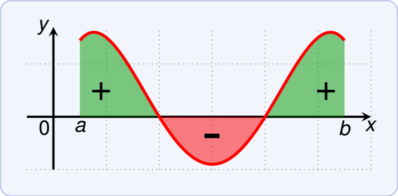
by interpreting it in terms of the area.



Thus the area is $1/4$ of the area of a circle with radius 1:

$$\int_0^1 \sqrt{1-x^2}dx = \frac{\pi}{4}$$

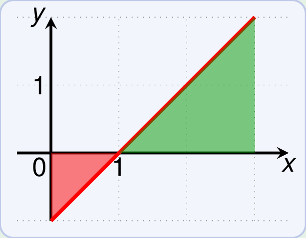
The Definite Integral



Evaluate the integral

$$\int_0^3 (x-1)\,dx$$

by interpreting it in terms of the area.



Thus the integral is:

$$\int_0^3 (x-1)\,dx = \frac{1}{2}(2\cdot 2) - \frac{1}{2}(1\cdot 1) = 1.5$$

The Definite Integral

The integral is a number.

The variable name x does not influence the integral:

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(r)dr$$

The Definite Integral

If the limit

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

exists, then f is called **integrable** on $[a, b]$.

Note every function is integrable.

However, most of the functions we work with are:

If

- f is continuous on $[a, b]$, or
- f has only a finite number of jump discontinuities,

then f is integrable on $[a, b]$, that is, the $\int_a^b f(x)dx$ exist.

The Definite Integral

If f is integrable on $[a, b]$, then the limit

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

gives the same value no matter how we choose the sample points x_i from the i-th interval.

Thus for simplicity we can choose the right end points.

This simplifies the definition of the definite integral:

If f is integrable on $[a, b]$, then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\quad \Delta x = \frac{b-a}{n} \quad$ and $\quad x_i = a + i\Delta$.

Computing with Sums

$$\sum_{i=1}^{n} c = nc$$

$$\sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

The Definite Integral

Evaluate the definite integral of f from 0 to 6 is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \qquad \text{where } f(x) = 2x - 5$$

using right endpoints for the sample points x_i.

Let $n > 0$. Then

- $\Delta x = (6-0)/n = 6/n$
- the i-th interval is $[0 + (i-1)\Delta x,\ 0 + i\Delta x]$
- the right endpoints are $x_i = i\Delta x$
- the values at x_i's are $f(x_i) = 2(i\Delta x) - 5$

The Definite Integral

Evaluate the definite integral of f from 0 to 6 is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \qquad \text{where } f(x) = 2x - 5$$

using right endpoints for the sample points x_i.

Let $n > 0$. Then $\Delta x = 6/n$ and $f(x_i) = 2(i\Delta x) - 5$.

Thus the Riemann sum is:

$$\begin{aligned}
R_n &= \sum_{i=1}^{n} f(x_i) \cdot \Delta x = \sum_{i=1}^{n} \left(2(i\Delta x) - 5\right) \Delta x \\
&= \Delta x \sum_{i=1}^{n} (2i\Delta x - 5) = \Delta x \left(\sum_{i=1}^{n} 2i\Delta x - \sum_{i=1}^{n} 5 \right) \\
&= \Delta x \left(2\Delta x \left(\sum_{i=1}^{n} i \right) - 5n \right) = \Delta x \left(2\Delta x \frac{n(n+1)}{2} - 5n \right)
\end{aligned}$$

The Definite Integral

Evaluate the definite integral of f from 0 to 6 is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \qquad \text{where } f(x) = 2x - 5$$

using right endpoints for the sample points x_i.

Let $n > 0$. Then $\Delta x = 6/n$ and $f(x_i) = 2(i\Delta x) - 5$.

Thus the Riemann sum is:

$$\begin{aligned} R_n &= \Delta x \left(2\Delta x \frac{n(n+1)}{2} - 5n \right) = \Delta x \left(\Delta x n(n+1) - 5n \right) \\ &= \frac{6}{n} \left(\frac{6}{n} n(n+1) - 5n \right) = \frac{6}{n} \left(6(n+1) - 5n \right) \\ &= \frac{6}{n} (n+1) = \frac{6n+6}{n} \end{aligned}$$

$$\int_a^b f(x)dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{6n+6}{n} = 6$$

Properties of the Definite Integral

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

$$\int_a^a f(x)dx = 0$$

$$\int_a^b c\;dx = c(b-a)$$

$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x) + \int_a^b g(x)$$

$$\int_a^b (f(x)-g(x))dx = \int_a^b f(x) - \int_a^b g(x)$$

$$\int_a^b cf(x)dx = c\int_a^b f(x)$$

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)$$

Properties of the Definite Integral

Assume $\int_0^{10} f(x)dx = 17$ and $\int_0^8 f(x)dx = 12$, find $\int_8^{10} f(x)dx$.

$$\int_0^8 f(x)dx + \int_8^{10} f(x)dx = \int_0^{10} f(x)dx \implies \int_8^{10} f(x)dx = 17 - 12 = 5$$

Use the properties of integrals to evaluate:

$$\begin{aligned} \int_0^1 (4 + 3x^2)dx &= \int_0^1 4\ dx + \int_0^1 3x^2 dx \\ &= 4 + 3\int_0^1 x^2 dx \\ &= 4 + 3\frac{1}{3} = 5 \end{aligned}$$

We have already seen that

$$\int_0^1 x^2 dx = \frac{1}{3}$$

Comparison Properties of the Definite Integral

- If $f(x) \geq 0$ for all $a \leq x \leq b$, then
$$\int_a^b f(x)dx \geq 0$$
- If $f(x) \geq g(x)$ for all $a \leq x \leq b$, then
$$\int_a^b f(x)dx \;\geq\; \int_a^b g(x)dx$$
- If $m \leq f(x) \leq M$ for all $a \leq x \leq b$, then
$$m(b-a) \;\leq\; \int_a^b f(x)dx \;\leq\; M(b-a)$$

Use the last property to estimate $\int_0^1 e^{-x^2}dx$.

The function e^{-x^2} is decreasing on $[0,1]$.

Thus on $[0,1]$: maximum is $f(0)=1$, and minimum $f(1)=e^{-1}$.

$$e^{-1}(1-0)=e^{-1} \quad\leq\quad \int_0^1 e^{-x^2}dx \quad\leq\quad 1=1(1-0)$$