MATH 211
Online Asynchronous Survey in Calculus and Analytical Geometry

Dr. Ahmed Kaffel

Department of Mathematical Sciences
University of Wisconsin Milwaukee

Spring 2023

Indefinite Integrals

The **indefinite integral**

$$\int f(x)dx$$

is a notation for an antiderivative. That is

$$\int f(x)dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)$$

For example

$$\int x^2\,dx = \frac{1}{3}x^3 + C$$

Note the difference between the definite and indefinite integral!

The definite integral $\int_a^b f(x)\,dx$ is a number.

The indefinite integral $\int f(x)\,dx$ is a function.

Indefinite Integrals

The **indefinite integral**

$$\int f(x)dx$$

is a notation for an antiderivative. That is

$$\int f(x)dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)$$

For example

$$\int x^2\,dx = \frac{1}{3}x^3 + C$$

The 2nd part of the Fundamental Theorem can be restated as:

If f is a continuous function then

$$\int_a^b f(x)\,dx \;=\; \int f(x)\,dx\Big]_a^b$$

Indefinite Integrals

Table of basic indefinite integrals:

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int k\,dx = kx + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad \int \frac{1}{x}dx = \ln|x| + C$$

$$\int e^x dx = e^x + C \qquad \int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

Indefinite Integrals

Find the general indefinite integral

$$\begin{aligned}\int(10x^4 - 2\sec^2 x)dx &= \int(10x^4)dx - \int(2\sec^2 x)dx \\ &= 10\int x^4 dx - 2\int \sec^2 x\, dx \\ &= 10\frac{1}{5}x^5 - 2\tan x + C \\ &= 2x^5 - 2\tan x + C\end{aligned}$$

Find the general indefinite integral

$$\begin{aligned}\int\left(\frac{\cos\phi}{\sin^2\phi}\right)d\phi &= \int\left(\frac{1}{\sin\phi}\cdot\frac{\cos\phi}{\sin\phi}\right)d\phi \\ &= \int(\csc\phi\cdot\cot\phi)\,d\phi \\ &= -\csc\phi + C\end{aligned}$$

Indefinite Integrals

Evaluate

$$\int_0^3 (x^3 - 6x)dx = (\frac{1}{4}x^4 - 3x^2)\Big]_0^3$$
$$= (\frac{1}{4}3^4 - 3\cdot 3^2) - (\frac{1}{4}0^4 - 3\cdot 0^2)$$
$$= \frac{81}{4} - 27$$

Evaluate

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2} dt = \int_1^9 (2 + \sqrt{t} - \frac{1}{t^2})dt$$
$$= (2t + \frac{2}{3}t^{\frac{3}{2}} + \frac{1}{t})\Big]_1^9$$
$$= (2\cdot 9 + \frac{2}{3}(\sqrt{9})^3 + \frac{1}{9}) - (2 + \frac{2}{3} + 1)$$
$$= 292/9$$

Indefinite Integrals: Applications

We can reformulate part 2 of the Fundamental Theorem as:

The integral of a rate of change is the net change.

$$\int_a^b F'(x)dx = F(b) - F(a)$$

Applications:

- $V(t)$ is the amount of water in a reservoir at time t
- $V'(t)$ is the rate at which water flows in or out

Then

$$\int_{t_1}^{t_2} V'(t)dt = V(t_2) - V(t_1)$$

is the net change in the amount of water from time t_1 to t_2.

Indefinite Integrals: Applications

We can reformulate part 2 of the Fundamental Theorem as:

The integral of a rate of change is the net change.

$$\int_a^b F'(x)dx = F(b) - F(a)$$

Applications:

- $C(t)$ is the concentration of a product of a chemical reation at time t
- $C'(t)$ is the rate of reaction

Then

$$\int_{t_1}^{t_2} C'(t)dt = C(t_2) - C(t_1)$$

is the change in concentration of C from time t_1 to t_2.

Indefinite Integrals: Applications

We can reformulate part 2 of the Fundamental Theorem as:

The integral of a rate of change is the net change.

$$\int_a^b F'(x)dx = F(b) - F(a)$$

Applications:

- $C(x)$ is the costs of producing x units of some product
- $C'(x)$ is the marginal costs

Then

$$\int_{x_1}^{x_2} C'(x)dx = C(x_2) - C(x_1)$$

is the increase in costs when the production is increased from x_1 to x_2.

Indefinite Integrals: Applications

We can reformulate part 2 of the Fundamental Theorem as:

The integral of a rate of change is the net change.

$$\int_a^b F'(x)dx = F(b) - F(a)$$

Applications:

We consider an object moving in a straight line:

- $s(t)$ is the position function
- $v(t) = s'(t)$ is the velocity

Then

$$\int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1)$$

is the net change of the position, the **displacement**, from time t_1 to t_2.

Indefinite Integrals: Applications

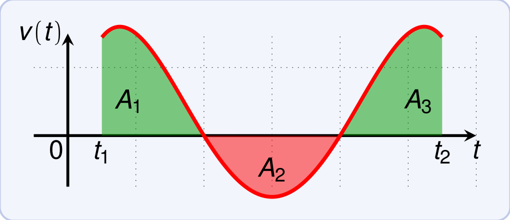
We consider an object moving in a straight line:

- $v(t)$ is the velocity

Then

$$\int_{t_1}^{t_2} |v(t)| dt$$

is the **total distance** the object traveled during the time interval.



displacement = $\int_{t_1}^{t_2} v(t)dt = A_1 - A_2 + A_3$

total distance = $\int_{t_1}^{t_2} |v(t)|dt = A_1 + A_2 + A_3$

Indefinite Integrals: Applications

We consider an object moving in a straight line:

- $a(t)$ is the acceleration
- $v(t)$ is the velocity

Then

$$\int_{t_1}^{t_2} a(t)dt = v(t_2) - v(t_1)$$

is the net change in velocity from t_1 to t_2.

Indefinite Integrals: Applications

We consider an object moving in a straight line with velocity

$$v(t) = t^2 - t - 6\text{m/s}$$

Find the displacement and distance traveled during $1 \leq t \leq 4$.

The displacement is

$$\begin{aligned}\int_1^4 v(t)dt &= (\frac{1}{3}v^3 - \frac{1}{2}t^2 - 6t)\Big]_1^4 \\ &= (\frac{1}{3}4^3 - \frac{1}{2}4^2 - 6\cdot 4) - (\frac{1}{3}1^3 - \frac{1}{2}1^2 - 6\cdot 1) \\ &= (\frac{64}{3} - \frac{16}{2} - 24) - (\frac{1}{3} - \frac{1}{2} - 6) = -\frac{9}{2}\end{aligned}$$

The displacement is -4.5m.

Indefinite Integrals: Applications

We consider an object moving in a straight line with velocity

$$v(t) = t^2 - t - 6\text{m/s}$$

Find the displacement and distance traveled during $1 \le t \le 4$.

The total distance traveled is $\int_1^4 |v(t)| dt$

We need to find the x-intercepts of $v(t)$ in $[1, 4]$:

$$v(t) = (t+2)(t-3) = 0 \quad \Longleftrightarrow \quad t = -2 \text{ or } t = 3$$

Thus we have:

$$\begin{aligned} \int_1^4 |v(t)| dt &= \left| \int_1^3 v(t) \right| + \left| \int_3^4 v(t) \right| \\ &= \left| (\frac{1}{3}v^3 - \frac{1}{2}t^2 - 6t)\Big]_1^3 \right| + \left| (\frac{1}{3}v^3 - \frac{1}{2}t^2 - 6t)\Big]_3^4 \right| \\ &= |-22/3| + |17/6| = 61/6 \end{aligned}$$

The total distance traveled is $61/6$ m.