MATH 211
Online Asynchronous Survey in Calculus and Analytical Geometry

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Spring 2023

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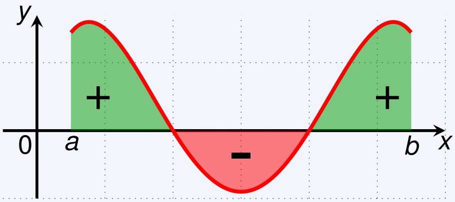
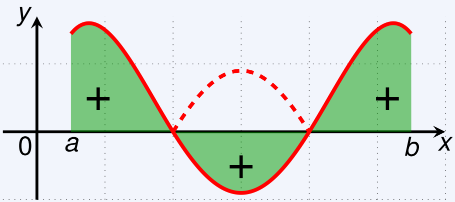
Area Between Curves

The definite integral can be interpreted as the **net area**, that is:



$$\int_a^b f(x)dx$$

What if we want the area between the curve and the x-axis?



$$\int_a^b |f(x)|dx$$

Area Between Curves

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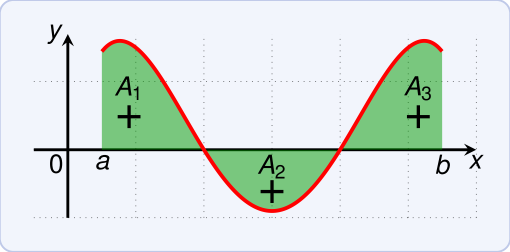

Let f be continuous on $[a, b]$.

Then the area between the curve f and the x-axis from a to b is

$$A = \int_a^b |f(x)| dx$$

To evaluate the integral, we split the it into A_1, A_2 and A_3.
Thus we must find the x-intercepts in $[a, b]$!

Area Between Curves

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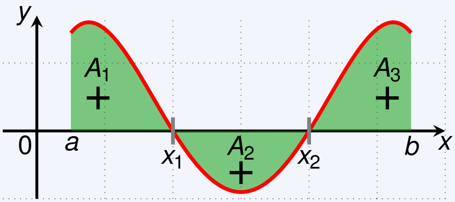


For example, let us consider the diagram above.

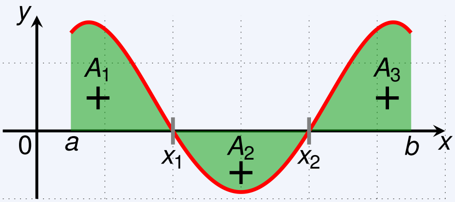
The area between the curve and the x-axis from a to b is

$$A = \int_a^b |f(x)| dx \; = \left| \int_a^{x_1} f(x)\, dx \right| + \left| \int_{x_1}^{x_2} f(x)\, dx \right| + \left| \int_{x_2}^{b} f(x)\, dx \right|$$

Note that we split the integral from a to the first x-intercept, from the first to the second x-intercept,...

Area Between Curves

What if we want the area between the curve and the x-axis?



Let f be continuous on $[a, b]$, and let

- $x_1 < x_2 < \ldots < x_n$ be all x-intercepts in $[a, b]$,
- define $x_0 = a$ and $x_{n+1} = b$

Then the area between the curve f and the x-axis from a to b is

$$A = \int_a^b |f(x)| dx \; = \; \sum_{i=0}^{n} \left| \int_{x_i}^{x_{i+1}} f(x)\, dx \right|$$

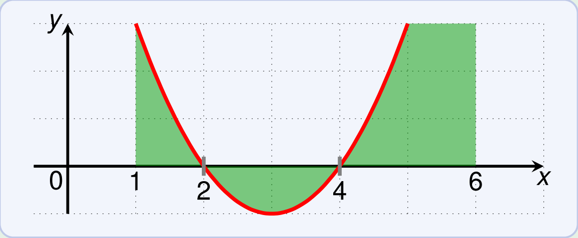
Area Between Curves

Find the area between $f(x) = x^2 - 6x + 8$ from 1 to 6.

An antiderivative of f is $F(x) = \frac{1}{3}x^3 - 3x^2 + 8x$.

We need to find the x-intercepts in $[1, 6]$:

$$f(x) = (x-2)(x-4) = 0 \quad \iff \quad x = 2 \text{ or } x = 4$$



Then the area between the curve and the x-axis from 1 to 6 is

$$A = \left|\int_1^2 f(x)dx\right| + \left|\int_2^4 f(x)dx\right| + \left|\int_4^6 f(x)dx\right|$$

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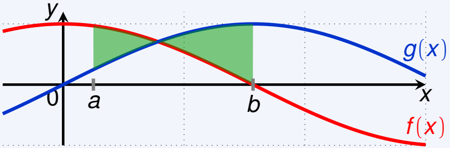
$$\begin{aligned}
A &= \left|\int_1^2 f(x)dx\right| + \left|\int_2^4 f(x)dx\right| + \left|\int_4^6 f(x)dx\right| \\
&= |F(2) - F(1)| + |F(4) - F(2)| + |F(6) - F(4)| \\
&= \left|\frac{20}{3} - \frac{16}{3}\right| + \left|\frac{16}{3} - \frac{20}{3}\right| + \left|\frac{36}{3} - \frac{16}{3}\right| \\
&= \left|\frac{4}{3}\right| + \left|-\frac{4}{3}\right| + \left|\frac{20}{3}\right| \\
&= \frac{28}{3}
\end{aligned}$$

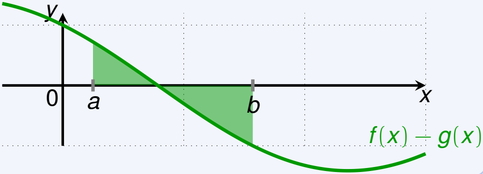
Area Between Curves

The area between two curves $f(x)$ and $g(x)$ from a to b is:

$$A = \int_a^b |f(x) - g(x)| dx$$



The area between f and g = area between $f - g$ and the x-axis.



Area Between Curves

Find the area of the region bounded by the curves

$$y = \sin x \qquad y = \cos x \qquad x = 0 \qquad x = \pi/2$$

Area is equal to the area between $\sin x - \cos x$ and the x-axis:

$$A = \int_0^{\pi/2} |\sin x - \cos x| dx$$

We have to find the x-intercepts in the interval $[0, \pi/2]$:

$$\sin x - \cos = 0 \iff \sin x = \cos x \iff x = \pi/4$$

Antiderivative of $f(x) = \sin x - \cos x$ is $F(x) = -\cos x - \sin x$:

$$A = \left|\int_0^{\pi/4} f(x)dx\right| + \left|\int_{\pi/4}^{\pi/2} f(x)dx\right| = |F(x)]_0^{\pi/4}| + |F(x)]_{\pi/4}^{\pi/2}|$$

$$= |(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}) - (-1 - 0)| + |(-0 - 1) - (-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}})|$$

$$= |-\frac{2}{\sqrt{2}} + 1| + |-1 + \frac{2}{\sqrt{2}}| = \sqrt{2} - 1 + -1 + \sqrt{2} = 2\sqrt{2} - 2$$

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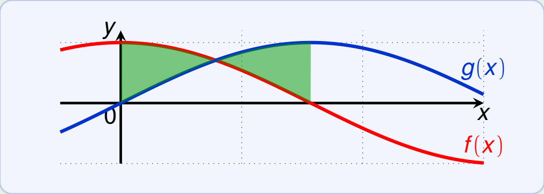

$$A = 2\sqrt{2} - 2$$