MATH 211
Online Asynchronous Survey in Calculus and Analytical Geometry

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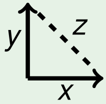
Practice problems

Two people are standing together. One begins to walk east at a rate of 2 miles per hour, and at the same time the second begins to walk north at a rate of 3 miles per hour. How fast is their distance growing when the first has walked 4 miles?



We know that

$$\frac{d}{dx} = 2 \qquad \frac{d}{dy} = 3 \qquad z^2 = x^2 + y^2$$

The first has walked 4 miles when $t = 4/2 = 2h$. At time $t = 2h$:

$$x = 4 \qquad y = 2 * 3 = 6 \qquad z = \sqrt{6^2 + 4^2} = \sqrt{52} = 2\sqrt{13}$$

We use implicit differentiation:

$$\frac{d}{dt} z^2 = \frac{d}{dt}(x^2 + y^2) \implies 2zz' = 2xx' + 2yy'$$

$$z' = \frac{2xx' + 2yy'}{2z} = \frac{2 \cdot 4 \cdot 2 + 2 \cdot 6 \cdot 3}{2 \cdot 2\sqrt{13}} = \frac{13}{\sqrt{13}} = \sqrt{13} \approx 3.6$$

Their distance increases with $\sqrt{13}$ miles per hour.

Practice problems

A bacteria culture is growing under ideal conditions and doubling every hour. If the initial population is 100 bacteria,

How many bacteria will there be after half an hour?

The formula for the population is of the form $P(t) = 100 \cdot e^{kt}$.

Let's determine k. After 1h we have 200 bacteria, thus

$$200 = 100 \cdot e^{k \cdot 1} \implies 2 = e^k \implies k = \ln 2$$

Thus $P(t) = 100 \cdot e^{t \ln 2} = 100 \cdot 2^t$.

After half an hour we have $100 \cdot 2^{\frac{1}{2}} \approx 141$ bacteria.

At what rate will the population be increasing at that point?

The rate of growth is $P'(t) = 100 \cdot \ln 2 \cdot 2^t$.

After half an hour the rate of growth is $100 \cdot \ln 2 \cdot 2^{\frac{1}{2}}$.

When will the bacteria population reach 1000?

$$1000 = 100 \cdot e^{t \ln 2} \iff 10 = e^{t \ln 2} \iff \ln 10 = t \ln 2$$

Thus after $t = \ln 10 / \ln 2 \approx 3.3$ hours.

Practice problems

Let $v(t)$ in m/s be the velocity of a particle moving along a line.

What does $\int_0^x v(t)dt$ tell us?

Net change of the position, thus the position after x seconds.

If $v(t) = t^2 - 3t + 2$, find the particles position after 1s.

An antiderivative of v is $V(t) = \frac{1}{3}t^3 - \frac{3}{2}t^2 + 2t$. Thus

$$p(1) = \int_0^1 v(t)dt = V(1) - V(0) = \frac{1}{3} - \frac{3}{2} + 2 = \frac{5}{6} \text{ m}$$

is the position of the particle after 1s.

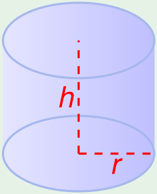
What is the average velocity during the first second?

The average velocity is

$$\frac{\Delta p}{\Delta t} = \frac{p(1) - p(0)}{1} = \frac{5}{6}\text{m/s}$$

Practice problems

A tin can is made to hold 1L of oil. Find the dimensions that minimize the cost of the metal to manufacture the can.



Introducing notation:

- let h be the height
- let r be the radius
- let V be the volume
- let A be the surface area

$$V = \pi r^2 h = 1 \implies h = 1/(\pi r^2)$$
$$A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2/r \qquad \text{for } r \text{ in } (0,\infty)$$
$$A'(r) = 4\pi r - 2/r^2 = (4\pi r^3 - 2)/r^2$$
$$A'(r) = 0 \iff r = 1/\sqrt[3]{2\pi} \text{ is the only critical number}$$

Cannot use Closed Interval Method since $(0,\infty)$ is not closed.

However, $A(1/\sqrt[3]{2\pi})$ must be the **absolute minimum** since:

- A is decreasing, $A'(r) < 0$, for all $r < 1/\sqrt[3]{2\pi}$,
- A is increasing, $A'(r) > 0$, for all $r > 1/\sqrt[3]{2\pi}$.

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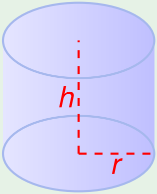

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Cannot use Closed Interval Method since $(0, \infty)$ is not closed.

However, $A(1/\sqrt[3]{2\pi})$ must be the **absolute minimum**

Then $h = 1/(\pi r^2) = \sqrt[3]{2\pi}^2/\pi = \sqrt[3]{4\pi^2/\pi^3} = 2/\sqrt[3]{2\pi} = 2r$

Hence radius $r = 1/\sqrt[3]{2\pi}$ and height $h = 2r$ minimizes the cost.

Practice problems

Evaluate the limit

$$\lim_{x \to 0^+} \frac{\cos x}{x}$$

When x approaches 0 from the right, we have

- $\cos x = 1$
- x is a small positive number

Thus

$$\lim_{x \to 0^+} \frac{\cos x}{x} = \infty$$

Practice problems

Find the area caught between $f(x) = x^2 - 2$ and the x-axis from $x = 1$ to $x = 2$.

The area corresponds to the following integral

$$A = \int_1^2 |f(x)|$$

To evaluate this integral, we need the x-intercepts in $[1, 2]$:

$$f(x) = 0 \iff x = \pm\sqrt{2}$$

Only $\sqrt{2}$ is in $[1, 2]$. Hence $A = \left|\int_1^{\sqrt{2}} f(x)\right| + \left|\int_{\sqrt{2}}^2 f(x)\right|$

An antiderivative of $f(x)$ is $F(x) = \frac{1}{3}x^3 - 2x$.

$$\begin{aligned} A &= |F(\sqrt{2}) - F(1)| + |F(2) - F(\sqrt{2})| \\ &= -(F(\sqrt{2}) - F(1)) + (F(2) - F(\sqrt{2})) = \frac{8\sqrt{2}}{3} - 3 \end{aligned}$$

Practice problems

Find the area of the region bounded by the curves

$$y = \sin x \qquad y = \cos x \qquad x = 0 \qquad x = \pi/2$$

Area is equal to the area between $\sin x - \cos x$ and the x-axis:

$$A = \int_0^{\pi/2} |\sin x - \cos x| dx$$

We have to find the x-intercepts in the interval $[0, \pi/2]$:

$$\sin x - \cos = 0 \iff \sin x = \cos x \iff x = \pi/4$$

Antiderivative of $f(x) = \sin x - \cos x$ is $F(x) = -\cos x - \sin x$:

$$\begin{aligned} A &= \left| \int_0^{\pi/4} f(x) dx \right| + \left| \int_{\pi/4}^{\pi/2} f(x) dx \right| = |F(x)]_0^{\pi/4}| + |F(x)]_{\pi/4}^{\pi/2}| \\ &= |(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}) - (-1 - 0)| + |(-0 - 1) - (-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}})| \\ &= | - \frac{2}{\sqrt{2}} + 1| + | - 1 + \frac{2}{\sqrt{2}}| = \sqrt{2} - 1 + -1 + \sqrt{2} = 2\sqrt{2} - 2 \end{aligned}$$

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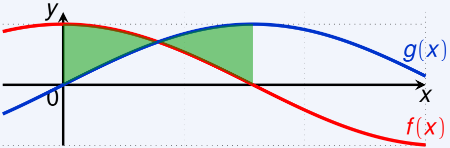

$A = 2\sqrt{2} - 2$

Practice problems

Consider the curve:

$$x^2 + y^2 = 1$$

At what point in the first quadrant has the curve slope -1?

We use implicit differentiation:

$$\begin{aligned} \frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}1 \\ 2x + 2yy' &= 0 \\ y' &= -\frac{x}{y} \end{aligned}$$

We know $x^2 + y^2 = 1$ and $y > 0$, thus $y = \sqrt{1 - x^2}$.

$$-1 = y' = -\frac{x}{\sqrt{1 - x^2}} \implies \sqrt{1 - x^2} = x \implies 1 - x^2 = x^2$$
$$\implies 2x^2 = 1 \implies x = \sqrt{1/2}$$

Thus the slope is -1 at point $(\sqrt{1/2}, \sqrt{1/2})$.

Practice problems

Find $\frac{dy}{dx}$ for the curve

$$\sin(x+y) = y^2 \cos x$$

We use implicit differentiation:

$$\begin{aligned}
&\frac{d}{dx}(\sin(x+y)) = \frac{d}{dx}(y^2 \cos x) \\
&\implies \cos(x+y)(1+y') = 2yy'\cos(x) + y^2(-\sin x) \\
&\implies y'\cos(x+y) - 2yy'\cos(x) = -y^2 \sin x - \cos(x+y) \\
&\implies y'(\cos(x+y) - 2y\cos(x)) = -y^2 \sin x - \cos(x+y) \\
&\implies \frac{dy}{dx} = y' = \frac{y^2 \sin x + \cos(x+y)}{2y\cos(x) - \cos(x+y)}
\end{aligned}$$

Practice problems

We consider the function $f(x)$ with

$$f(x) = \frac{1 - x^3}{1 + x^3} \quad f'(x) = \frac{6x^2}{(1 + x^3)^2} \quad f''(x) = \frac{12x(2x^3 - 1)}{(1 + x^3)^3}$$

Find all

- horizontal, vertical asymptotes,
- the left and right limits at vertical asymptotes,
- points with horizontal tangents and local extrema
- on which intervals is f increasing/decreasing?
- on which intervals is f concave up/down?
- inflection points

Then sketch the graph of $f(x)$.

Practice problems

We consider the function $f(x)$ with

$$f(x) = \frac{1 - x^3}{1 + x^3} \quad f'(x) = \frac{6x^2}{(1 + x^3)^2} \quad f''(x) = \frac{12x(2x^3 - 1)}{(1 + x^3)^3}$$

Practice problems

What are the critical numbers of

$$f(x) = x^{3/5}(x-5)$$

First, we simplify

$$f(x) = x^{8/5} - 5x^{3/5}$$
$$f'(x) = \frac{8}{5}x^{3/5} - 3x^{-\frac{2}{5}} = x^{3/5}\left(\frac{8}{5} - \frac{3}{x}\right)$$

So the critical numbers are

- $x = 0$, then $f'(x)$ undefined
- $x = \frac{15}{8}$, then $f'(x) = 0$

Practice problems

Show that the equation

$$e^x = x^3 + 1$$

has a solution for x in the real numbers. We define

$$f(x) = e^x - x^3 - 1$$

Then $f(x) = 0 \iff x$ is a solution of the equation.

We have

$$f(-1) = e^{-1} - -1^3 - 1 = e^{-1} - 2 < 0$$
$$f(1) = e^1 - 1^3 - 1 = e^1 > 0$$

More over f is continuous!

Thus by the Intermediate Value Theorem there exists c in $[-1, 1]$ such that $f(c) = 0$.

Hence the equation has a solution $x = c$.