MATH 211
Online Asynchronous Survey in Calculus and Analytical Geometry

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The Substitution Rule for Indefinite Integrals

How to find antiderivatives of formulas like

$$\int 2x\sqrt{1+x^2}dx \quad ?$$

Recall the chain rule:

$$\big(f(g(x))\big)' = f'(g(x))g'(x)$$

Let us try to write the integral in the form:

$$\int f'(g(x))g'(x)dx$$

Then

$$f'(x) = \sqrt{x} \qquad g(x) = 1 + x^2 \qquad g'(x) = 2x$$

Moreover the antiderivative of $f'(x)$ is $f(x) = \frac{2}{3}x^{\frac{3}{2}}$. Thus

$$\int 2x\sqrt{1+x^2}dx = f(g(x)) + C = \frac{2}{3}(1+x^2)^{\frac{3}{2}} + C$$

The Substitution Rule for Indefinite Integrals

Substitution Rule

If $u = g(x)$ is differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

To remember this rule: note that if $u = g(x)$, then

$$du = g'(x)dx$$

(here we think of dx and du as differentials)

In other words:

$$dx = \frac{du}{g'(x)}$$

If we change the variable from x to $u = g(x)$ we divide by $g'(x)$!

The Substitution Rule for Indefinite Integrals

Substitution Rule

If $u = g(x)$ is differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

If we change the variable from x to $u = g(x)$ we divide by $g'(x)$!

$$\int 2x\sqrt{1+x^2}dx$$

We choose $u = 1 + x^2$. Then $u' = 2x$, and hence

$$\int 2x\sqrt{1+x^2}dx = \int 2x\sqrt{u}\,\frac{du}{2x} = \int \sqrt{u}\,du$$

$$= \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(1+x^2)^{\frac{3}{2}} + C$$

The Substitution Rule for Indefinite Integrals

Substitution Rule

If $u = g(x)$ is differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

If we change the variable from x to $u = g(x)$ we divide by $g'(x)$!

$$\int x^3 \cos(x^4 + 2)dx$$

We choose $u = x^4 + 2$. Then $u' = 4x^3$, and hence

$$\int x^3 \cos(x^4 + 2)dx = \int x^3 \cos(u)\,\frac{du}{4x^3} = \frac{1}{4}\int \cos(u)\,du$$

$$= \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^4 + 2) + C$$

Finding the right u is a guessing game. Often multiple choices.

The Substitution Rule for Indefinite Integrals

$$\int \sqrt{2x+1}dx$$

We choose $u = 2x + 1$. Then $u' = 2$, and hence

$$\int \sqrt{2x+1}dx = \int \sqrt{u}\frac{du}{2} = \frac{1}{2}\int \sqrt{u}\,du$$
$$= \frac{1}{2} \cdot \frac{2}{3}u^{\frac{3}{2}} + C = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C$$

We could have chosen another u. For example:

We choose $u = \sqrt{2x+1}$. Then $u' = \frac{1}{2\sqrt{2x+1}} \cdot 2 = \frac{1}{u}$. Then

$$\int \sqrt{2x+1}dx = \int u\,\frac{du}{1/u} = \int u^2\,du$$
$$= \frac{1}{3}u^3 + C = \frac{1}{3}\left(\sqrt{2x+1}\right)^3 + C = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C$$

The Substitution Rule for Indefinite Integrals

$$\int \frac{x}{\sqrt{1-4x^2}} dx$$

We choose $u = 1 - 4x^2$. Then $u' = -8x$, and hence

$$\int \frac{x}{\sqrt{1-4x^2}} dx = \int \frac{x}{\sqrt{u}} \frac{du}{-8x} = -\frac{1}{8} \int \frac{1}{\sqrt{u}} du$$

$$= -\frac{1}{8} \cdot 2\sqrt{u} + C = -\frac{1}{4}\sqrt{1-4x^2} + C$$

$$\int e^{5x} dx$$

We choose $u = 5x$. Then $u' = 5$, and hence

$$\int e^{5x} dx = \int e^u \frac{du}{5} = \frac{1}{5} \int e^u du$$

$$= \frac{1}{5} e^u + C = \frac{1}{5} e^{5x} + C$$

The Substitution Rule for Indefinite Integrals

A slightly more interesting example:

$$\int x^5\sqrt{1+x^2}\,dx$$

We choose $u = 1 + x^2$. Then $u' = 2x$, and hence

$$\int x^5\sqrt{1+x^2}\,dx = \int x^5\sqrt{u}\,\frac{du}{2x} = \frac{1}{2}\int x^4\sqrt{u}\,du$$

What now? Note that $\quad x^2 = u - 1 \quad$ and $\quad x^4 = (x^2)^2$

$$\int x^5\sqrt{1+x^2}\,dx = \frac{1}{2}\int x^4\sqrt{u}\,du = \frac{1}{2}\int (u-1)^2\sqrt{u}\,du$$

$$= \frac{1}{2}\int (u^2 - 2u + 1)\sqrt{u}\,du = \frac{1}{2}\int \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)du$$

$$= \frac{1}{2}\left(\frac{2}{7}u^{\frac{7}{2}} - 2\cdot\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right) + C$$

$$= \frac{1}{7}(1+x^2)^{\frac{7}{2}} - \frac{2}{5}(1+x^2)^{\frac{5}{2}} + \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C$$

The Substitution Rule for Indefinite Integrals

$$\int \tan x \, dx$$

First, we note that:

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

We choose $u = \cos x$. Then $u' = -\sin x$, and hence

$$\int \frac{\sin x}{\cos x} dx = \int \frac{\sin x}{u} \frac{du}{-\sin x} = -\int \frac{1}{u} \, du$$
$$= -\ln|u| + C = -\ln|\cos x| + C$$

Note that

$$-\ln|\cos x| = \ln|\cos x|^{-1} = \ln|\sec x|$$

Thus we can also write:

$$\int \tan x \, dx = \ln|\sec x| + C$$

The Substitution Rule for Definite Integrals

Methods for evaluating a **definite integral using substitution**:

$$\int_a^b f(x)dx$$

Method 1

- evaluate the indefinite integral $\int f(x)dx$ using substitution
- then use the Fundamental Theorem $\int_a^b f(x)dx = \int f(x)dx\Big]_a^b$

$$\int_0^4 \sqrt{2x+1}\,dx = \int \sqrt{2x+1}\,dx\Big]_0^4 = \frac{1}{3}(2x+1)^{\frac{3}{2}}\Big]_0^4 = 9 - \frac{1}{3} = \frac{26}{3}$$

Method 2: Substitution Rule for Definite Integrals

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

The Substitution Rule for Definite Integrals

Method 2: Substitution Rule for Definite Integrals

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

$$\int_0^4 \sqrt{2x+1}\,dx$$

We choose $u = 2x + 1$. Then $u' = 2$, and hence

$$\begin{aligned}\int_0^4 \sqrt{2x+1}\,dx &= \int_{u(0)}^{u(4)} \sqrt{u}\,\frac{du}{2} = \frac{1}{2}\int_1^9 \sqrt{u}\,du \\ &= \frac{1}{2}(\frac{2}{3}u^{\frac{3}{2}})\Big]_1^9 = \frac{1}{2}(\frac{2}{3}9^{\frac{3}{2}} - \frac{2}{3}1^{\frac{3}{2}}) \\ &= \frac{1}{2}(\frac{2}{3}\sqrt{9}^3 - \frac{2}{3}) = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}\end{aligned}$$

The Substitution Rule for Definite Integrals

Method 2: Substitution Rule for Definite Integrals

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_{a}^{b} f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

$$\int_{1}^{2} \frac{1}{(3-5x)^2}\,dx$$

We choose $u = 3 - 5x$. Then $u' = -5$, and hence

$$\int_{1}^{2} \frac{1}{(3-5x)^2}\,dx = \int_{u(1)}^{u(2)} \frac{1}{u^2}\frac{du}{-5} = -\frac{1}{5}\int_{-2}^{-7} \frac{1}{u^2}\,du$$

$$= -\frac{1}{5}\left(-\frac{1}{u}\right)\Big]_{-2}^{-7} = -\frac{1}{5}\left(-\frac{1}{-7} - \left(-\frac{1}{-2}\right)\right)$$

$$= -\frac{1}{5}(\frac{1}{7} - \frac{1}{2}) = -\frac{1}{5}\left(\frac{2}{14} - \frac{7}{14}\right) = \frac{1}{14}$$

The Substitution Rule for Definite Integrals

Method 2: Substitution Rule for Definite Integrals

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_{a}^{b} f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

$$\int_{1}^{e} \frac{\ln x}{x}\,dx$$

We choose $u = \ln x$. Then $u' = \frac{1}{x}$, and hence

$$\int_{1}^{e} \frac{\ln x}{x}\,dx = \int_{u(1)}^{u(e)} \frac{u}{x}\,\frac{du}{1/x} = \int_{0}^{1} u\,du$$

$$= \left(\frac{1}{2}u^2\right)\Big]_0^1 = \frac{1}{2}1^2 - \frac{1}{2}0^2 = \frac{1}{2}$$

The Substitution Rule: Exercises

$$\int e^{-x}\,dx \qquad \text{take } u = -x$$

$$\int x^3(2+x^4)^5\,dx \qquad \text{take } u = 2+x^4$$

$$\int x^2\sqrt{x^3+1}\,dx \qquad \text{take } u = x^3+1$$

$$\int \frac{1}{(1-6t)^4}\,dt \qquad \text{take } u = 1-6t$$

$$\int \cos^3\phi\ \sin\phi\,dt \qquad \text{take } u = \cos\phi$$

$$\int \frac{\sec^2(\frac{1}{x})}{x^2}\,dt \qquad \text{take } u = \frac{1}{x}$$

The Substitution Rule: Exercises

$$\int (x+1)\sqrt{2x+x^2}\,dx \quad \text{take } u = 2x + x^2\text{, then } u' = 2(1+x)$$

$$\int (3t+2)^{2.4}\,dx \qquad \text{take } u = 3t+2$$

$$\int e^x \cos e^x\,dx \qquad \text{take } u = e^x$$

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dt \qquad \text{take } u = \sqrt{x}\text{, then } u' = \frac{1}{2\sqrt{x}}$$

$$\int \frac{(\ln x)^2}{x}\,dt \qquad \text{take } u = \ln x$$

$$\int (x^3+3x)(x^2+1)\,dt \quad \text{take } u = x^3+3x\text{, then } u' = 3(x^2+1)$$

The Substitution Rule: Exercises

Evaluate

$$\int x \sin(x^2)\, dx$$

We take $u = x^2$, then $u' = 2x$ and

$$\begin{aligned}
\int x \sin(x^2)\, dx &= \int x \sin u \, \frac{du}{2x} \\
&= \frac{1}{2} \int \sin u \, du \\
&= -\frac{1}{2} \cos u + C \\
&= -\frac{1}{2} \cos x^2 + C
\end{aligned}$$

The Substitution Rule: Exercises

Evaluate

$$\int x^2 e^{x^3}\,dx$$

We take $u = x^3$, then $u' = 3x^2$ and

$$\begin{aligned}\int x^2 e^{x^3}\,dx &= \int x^2 e^u \,\frac{du}{3x^2}\\ &= \frac{1}{3}\int e^u\,du\\ &= \frac{1}{3}e^u + C\\ &= \frac{1}{3}e^{x^3} + C\end{aligned}$$

The Substitution Rule: Exercises

Evaluate

$$\int \frac{\sin 2x}{1+\cos^2 x}\,dx$$

We recall that $\quad \sin 2x = 2\sin x \cos x$

We take $u = 1 + (\cos x)^2$.

Then $u' = 2\cos x(-\sin x) = -\sin 2x$ and

$$\begin{aligned}\int \frac{\sin 2x}{1+\cos^2 x}\,dx &= \int \frac{\sin 2x}{u}\,\frac{du}{-\sin 2x}\\ &= -\int \frac{1}{u}\,du\\ &= -\ln|u| + C\\ &= -\ln|1+(\cos x)^2| + C\end{aligned}$$

The Substitution Rule: Exercises

Evaluate

$$\int_0^1 \cos(\pi x/2)\,dx$$

We take $u = \pi x/2$, then $u' = \pi/2$ and

$$\begin{aligned}\int_0^1 \cos(\pi x/2)\,dx &= \int_{u(0)}^{u(1)} \cos(u)\,\frac{du}{\pi/2}\\ &= \frac{2}{\pi}\int_0^{\pi/2} \cos(u)\,du\\ &= \frac{2}{\pi}\left(\sin u\right]_0^{\pi/2}\Big)\\ &= \frac{2}{\pi}\left(\sin(\pi/2) - \sin(0)\right)\\ &= \frac{2}{\pi}\end{aligned}$$

Symmetry

Symmetry can sometimes help to simplify integrals!

Suppose f is continuous on $[-a, a]$:

- If f is even $[f(-x) = f(x)]$, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

- If f is odd $[f(-x) = -f(x)]$, then

$$\int_{-a}^{a} f(x)\,dx = 0$$

$$\int_{-1}^{1} f(x)\,dx = 0 \qquad \text{where} \qquad f(x) = \frac{\tan x}{1 + x^2 + x^4}$$

The function f is odd since

$$f(-x) = \frac{\sin(-x)/\cos(-x)}{1 + (-x)^2 + (-x)^4} = \frac{-\sin x/\cos x}{1 + x^2 + x^4} = -f(x)$$