MATH 211
Online Asynchronous Survey in Calculus and Analytical Geometry

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Average Value of a Function

We consider an object moving in a straight line:

- $s(t)$ the position at time t
- $v(t)$ the speed at time t

What is the average velocity over time interval $[a, b]$?

$$v_{\text{avg}} = \frac{s(b) - s(a)}{b - a}$$

By the Fundamental Theorem we have

$$s(b) - s(a) = \int_a^b v(t)dt$$

Hence

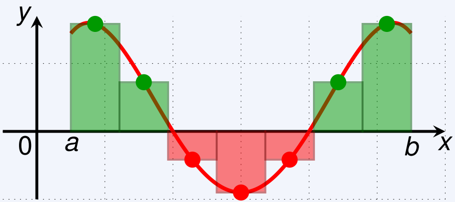
$$v_{\text{avg}} = \frac{1}{b - a}\int_a^b v(t)dt$$

The Mean Value Theorem tell us that:

There exists c in (a, b) such that $v(c) = v_{\text{avg}} = \frac{1}{b-a}\int_a^b v(t)dt$.

Average Value of a Function

How to compute the average value of a function?



Idea: split in n rectangles, take their average height.

$$\frac{f(x_1) + f(x_2) + \ldots + f(x_n)}{n} = \frac{1}{n}\sum_{i=1}^{n} f(x_i) = \frac{1}{n\,\Delta x}\sum_{i=1}^{n} f(x_i)\Delta x$$
$$= \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x$$

The sum $\sum_{i=1}^{n} f(x_i)\Delta x$ is called **Riemann sum.**

Average Value of a Function

If we let n go to infinity:

$$\lim_{n\to\infty} \frac{1}{b-a} \sum_{i=1}^{n} f(x_i)\Delta x$$

we get

$$\frac{1}{b-a} \int_a^b f(x)\, dx$$

As a consequence, we have

The average value $f_{\text{avg}}$of a function f on an interval $[a, b]$ is:

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x)\, dx$$

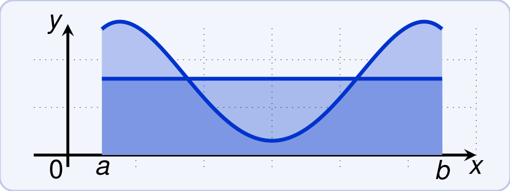
Thus the average value of the function is the integral over the interval divided by the width of the interval.

Average Value of a Function

The average value f_{avg} of a function f on an interval $[a, b]$ is:

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

This is **easy to remember**:



- Think of the area below the function as water.
- Then the amount of water is $A = \int_a^b f(x)dx$
- When the waves calm, the water settles in the shape of a rectangle with area A and width $b-a$; thus height $\frac{A}{b-a}$

Average Value of a Function

The average value f_{avg} of a function f on an interval $[a, b]$ is:

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Find the average value of the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$.

$$\begin{aligned} f_{\text{avg}} &= \frac{1}{2-(-1)}\int_{-1}^{2} f(x)dx \\ &= \frac{1}{3}\left(x + \frac{1}{3}x^3\right)\Big]_{-1}^{2} \\ &= \frac{1}{3}\left(2 + \frac{1}{3}2^3 - \left((-1) + \frac{1}{3}(-1)^3\right)\right) \\ &= \frac{1}{3}\left(2 + \frac{8}{3} + 1 + \frac{1}{3}\right) = 2 \end{aligned}$$

Average Value of a Function

Let F be an antiderivative of f. We know that

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}(F(b) - F(a))$$

By the Mean Value Theorem there exists c in $[a, b]$ such that

$$f(c) = \frac{1}{b-a}(F(b) - F(a))$$

Thus we obtain:

Mean Value Theorem for Integrals

If f is continuous an $[a, b]$, then there exists a number c in $[a, b]$ such that:

$$f(c) = f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

that is,

$$f(c)(b-a) = \int_a^b f(x)\,dx$$

Average Value of a Function

Mean Value Theorem **for Integrals**

If f is continuous an $[a, b]$, then there is c in $[a, b]$ such that:

$$f(c) \;=\; f_{\text{avg}} \;=\; \frac{1}{b-a}\int_a^b f(x)\,dx$$

Since $f(x) = 1 + x^2$ is continuous on $[-1, 2]$, the Mean Value Theorem for Integrals says...

There is a number c in $[-1, 2]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

For this f, we can find c explicitly. Since $\frac{1}{b-a}\int_a^b f(x)\,dx = 2$

$$2 = f(c) = 1 + c^2 \implies c^2 = 1 \implies c = \pm 1$$

Thus there are two numbers $c = -1$ and $c = 1$ that work!