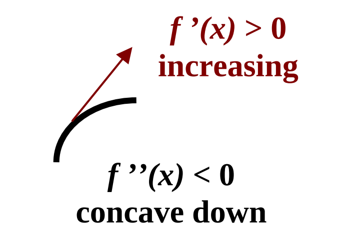
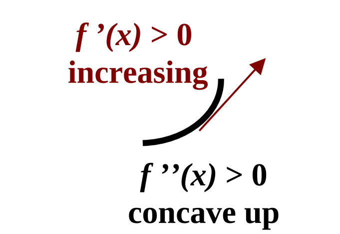
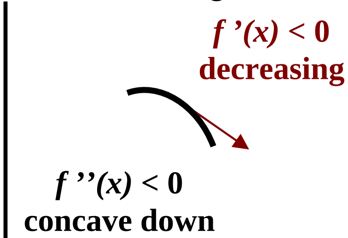
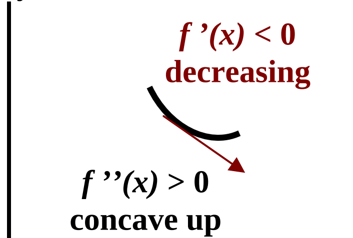
Using the Derivative to Analyze Functions

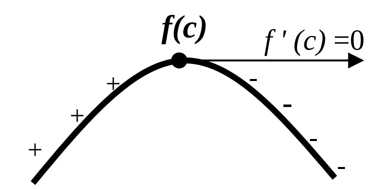
- $f'(x)$ indicates if the function is: Increasing or Decreasing on certain intervals.
  Critical Point c is where $f'(c) = 0$ *(tangent line is horizontal)*, or $f'(c)$ = undefined *(tangent line is vertical)*
- $f''(x)$ indicates if the function is concave up or down on certain intervals.
  Inflection Point: where $f''(x) = 0$ or where the function changes concavity, no Min no Max.



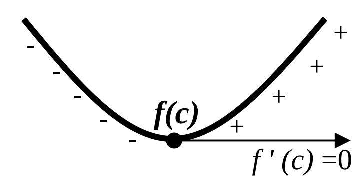




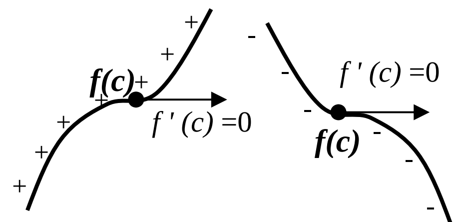




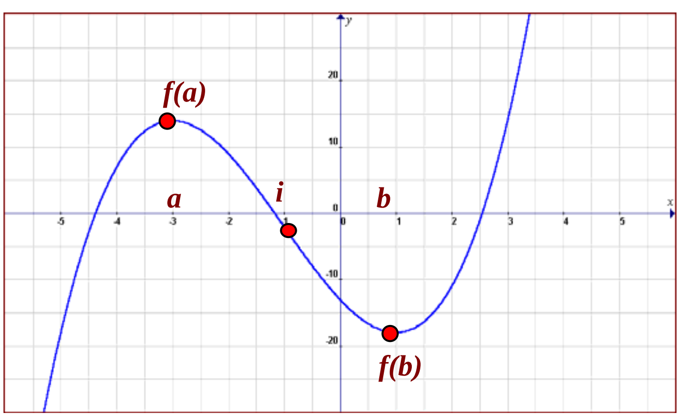
If the sign of $f'(c)$ changes:
from + to - , then:
$f(c)$ is a **local Maximum**



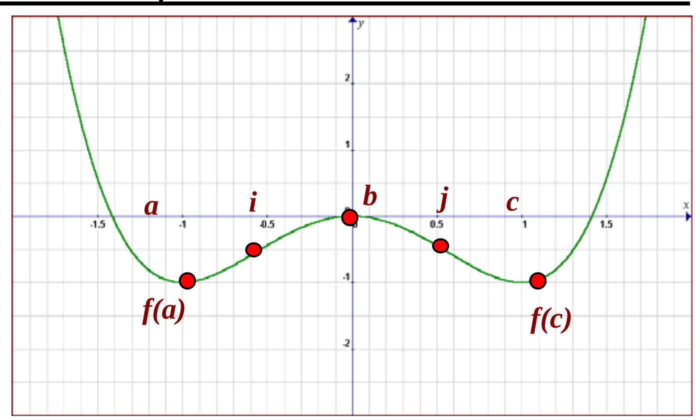
If the sign of $f'(c)$ changes:
from - to + , then:
$f(c)$ is a **local Minimum**



If there is no sign change for $f'(c)$:
then $f(c)$ is **not** a local extreme, it is:
An **Inflection Point** (*concavity changes*)



- **Critical points, $f'(x) = 0$** at : $x = a$, $x = b$
- **Increasing, $f'(x) > 0$** in : $x < a$ and $x > b$
- **Decreasing, $f'(x) < 0$** in : $a < x < b$
- **Max** at: $x = a$, **Max** = $f(a)$
- **Min** at: $x = b$, **Min** = $f(b)$
- **Inflection point, $f''(x) = 0$** at : $x = i$
- **Concave up, $f''(x) > 0$** in: $x > i$
- **Concave Down, $f''(x) < 0$** in: $x < i$



- **Critical points, $f'(x) = 0$** at : $x = a, x = b, x = c$
- **Increasing, $f'(x) > 0$** in : $a < x < b$ and $x > c$
- **Decreasing, $f'(x) < 0$** in : $x < a$ and $b < x < c$
- **Max** at: $x = b$, **Max** = $f(b)$
- **Min** at: $x = a$, $x = c$, **Min** = $f(a)$ and $f(c)$
- **Inflection point, $f''(x) = 0$** at : $x = i, x = j$
- **Concave up, $f''(x) > 0$** in: $x < i$ and $x > j$
- **Concave Down, $f''(x) < 0$** in: $i < x < j$

I) Applications of The First Derivative:

- Finding the critical points
- Determining the intervals where the function is increasing or decreasing
- Finding the local maxima and local minima

- **Step 1:** Locate the **critical points** where the derivative is = 0;
  find $f\,'(x)$ and make it = 0
  $f\,'(x) = 0 \quad \Rightarrow \; x = a, b, c, \ldots\ldots$

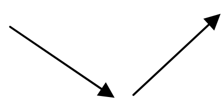
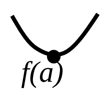
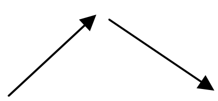
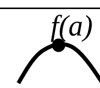
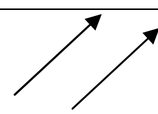
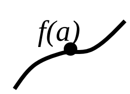
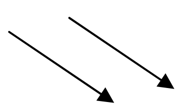
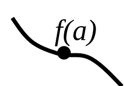
- **Step 2:** Divide $f\,'(x)$ into intervals using the critical points found in the previous step:

  then choose a **test point** in each interval.

- **Step 3:** Find the derivative for the function in each test point:

Sign of f ' **(test point)**	**Label the interval of the test point:**
> 0 or positive	*increasing* , + + + + + ,
< 0 or negative	*decreasing* , - - - - - - - ,

- **Step 4:** Look at both sides of each critical point, take point ***a*** for example:

- - - - - - - - - a + + + + + +	or		then it is a local **Min.**		**Min** = $f(a)$
+ + + + + + a - - - - - - - - -	or		then it is a local **Max.**		**Max** = $f(a)$
+ + + + + + a + + + + + +	or		**No** local Min. or Max**.**		
- - - - - - - - - a - - - - - - - - -	or		**No** local Min. or Max.		

II) Applications of The Second Derivative:

- Finding the inflection points
- Determining the intervals where the function is concave up or concave down

- **Step 5:** Locate the **inflection points** where the second derivative is = 0;
  find $f''(x)$ and make it = 0
  $f''(x) = 0 \quad \Rightarrow \; x = i, j, k, \ldots\ldots$

- **Step 6:** Divide $f''(x)$ into intervals using the inflection points found in the previous step:

  then choose a **test point** in each interval.

- **Step 7:** Find the second derivative for function in each test point:

Sign of ***f ''*** **(test point)**	**Label the interval of the test point:**
> 0 or positive	*Concave up* , + + + + + +, 
< 0 or negative	*Concave down* , - - - - - - - , 

- **Step 8:** Summarize all results in the following table:

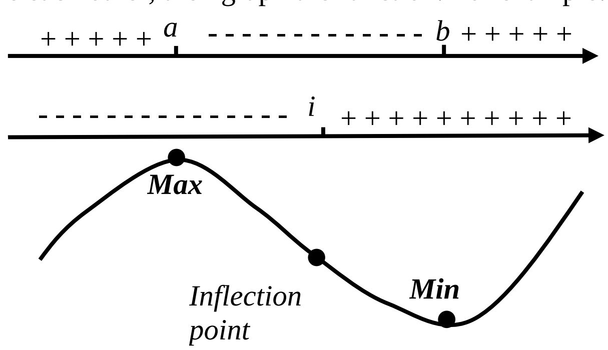
Increasing in the intervals:	
Decreasing in the intervals:	
Local Max. points and Max values:	
Local Min. points and Min values:	
Inflection points at:	
Concave Up in the intervals:	
Concave Down in the intervals:	

- **Step 9:** Sketch the graph using the information from **steps 3,4** and **7** showing the critical points, inflection points, intervals of increasing or decreasing, local maxima and minima and the intervals of concave up or down.

Note: It is best to put the data from steps 3,4,7 above each other, then graph the function. For example:

Steps 3,4: $f'(x)$, increasing, decreasing labels:

Step 7: $f''(x)$, concave up, down labels:



Show the coordinates of each point:
Local Max at **(*a*, *f(a)*)**
Local Min at **(*b*, *f(b)*)**
Inflection Point at **(*i* , *f(i)*)**

Example 1: For the function $f(x) = -x^3 + 3x^2 - 4$:

a) Find the intervals where the function is increasing, decreasing.
b) Find the local maximum and minimum points and values.
c) Find the inflection points.
d) Find the intervals where the function is concave up, concave down.
e) Sketch the graph

I) Using the First Derivative:

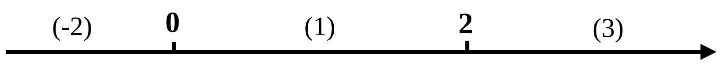
- **Step 1**: Locate the **critical points** where the derivative is = 0:

$$f'(x) = -3x^2 + 6x$$

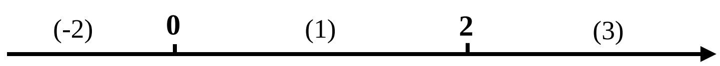
$$f'(x) = 0 \text{ then } \quad 3x(x - 2) = 0.$$

Solve for x and you will find $\mathbf{x = 0}$ and $\mathbf{x = 2}$ as the critical points

- **Step 2:** Divide $f'(x)$ into intervals using the critical points found in the previous step, then choose a **test points** in each interval such as (-2), (1), (3).

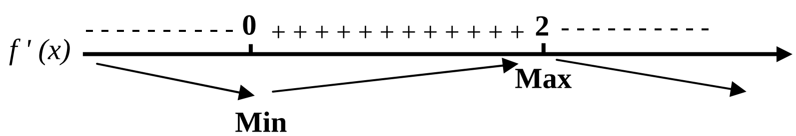


- **Step 3:** Find the derivative for the function in each test point: *(It is recommended to create a table underneath)*



$f'(x) = -3x^2 + 6x$	$f'(-2) = -24$	$f'(1) = +3$	$f'(3) = -9$
Sign	- - - - - - - -	+ + + + + + + + + +	- - - - - - - - - - - -
Shape	**Decreasing**	**Increasing**	**Decreasing**
Intervals	$\mathbf{x < 0}$	$\mathbf{0 < x < 2}$	$\mathbf{x > 2}$

- **Step 4:** Look at both sides of each critical point:



Local Minimum at $x = 0$, Minimum $= f(0) = -(0)^3 + 3(0)^2 - 4 = -4$; or **Min (0 , -4)**

Local Maximum at $x = 2$, Maximum $= f(2) = -(2)^3 + 3(2)^2 - 4 = 0$; or **Max (2 , 0)**

Increasing or $f'(x) > 0$ in: $0 < x < 2$

Decreasing or $f'(x) < 0$ in: $x < 0$ and $x > 2$

Example 1, continue

II) Using the Second Derivative:

- **Step 5:** Locate the **inflection points** where the second derivative is = 0; find $f''(x)$ and make it = 0
  $f'(x) = -3x^2 + 6x$
  $f''(x) = -6x + 6$
  $f''(x) = 0$ then $-6x + 6 = 0$
  Solve for x and you will find $\mathbf{x = 1}$ as the inflection point
- **Step 6:** Divide $f''(x)$ into intervals using the inflection points found in the previous step, then choose a **test point** in each interval such as (0) and (2).

  (0) **1** (2)

- **Step 7:** Find the second derivative for the function in each test point: *(It is recommended to create a table underneath)*

  (0) **1** (2)

$f''(x) = -6x + 6$	$f''(0) = 6$	$f''(2) = -6$
Sign	+ + + + + + + + + + +	- - - - - - - - - - - - - - - - - -
Shape	**Concave up**	**Concave Down**
Intervals	$\mathbf{x < 1}$	$\mathbf{x > 1}$

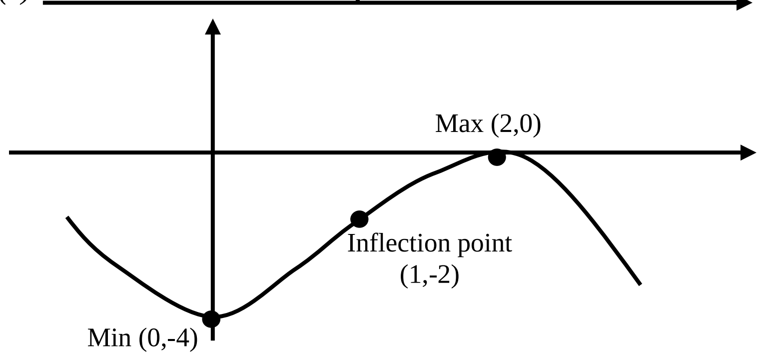
- **Step 8:** Summarize all results in the following table:

Increasing in the intervals:	$f'(x) > 0$ in $\mathbf{0 < x < 2}$
Decreasing in the intervals:	$f'(x) < 0$ in $\mathbf{x < 0}$ and $\mathbf{x > 2}$
Local Max. points and Max values:	Max. at $x = 2$, **Max (2 , 0)**
Local Min. points and Min values:	Min. at $x = 0$, **Min (0 , -4)**
Inflection points at:	$x = 1$, $f(1) = -2$ or at **(1,-2)**
Concave Up in the intervals:	$f''(x) > 0$ in $\mathbf{x < 1}$
Concave Down in the intervals:	$f''(x) < 0$ in $\mathbf{x > 1}$

- **Step 9:** Sketch the graph:

$f'(x)$ - - - - - - - - - - **0** + + + + + + + + + + + + **2** - - - - - - - - - -

$f''(x)$ + + + + + + + + + + + + + + **1** -

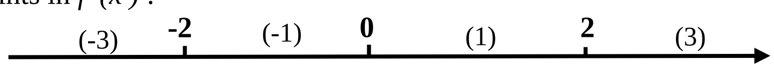


Example 2: Analyze the function $f(x) = 3x^5 - 20x^3$

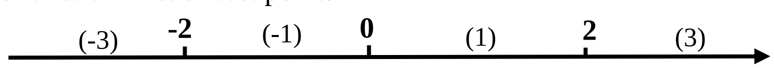
a) Find the intervals where the function is increasing, decreasing.
b) Find the local maximum and minimum points and values.
c) Find the inflection points.
d) Find the intervals where the function is concave up, concave down.
e) Sketch the graph

I) Using the First Derivative:

- **Step 1**: The **critical points** where the derivative is = 0:
  $f'(x) = 15x^4 - 60x^2$
  $f'(x) = 0$ then $15x^2(x^2 - 4) = 0.$
  Solve for x and you will find $\boldsymbol{x = -2}$, $\boldsymbol{x = 0}$ and $\boldsymbol{x = 2}$ as the critical points
- **Step 2:** Intervals & test points in $f'(x)$:

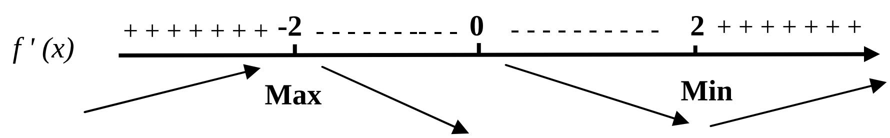


- **Step 3:** Derivative for the function in each test point:



$f'(x) = 15x^4 - 60x^2$	$f'(-3)= 675$	$f'(-1)= -45$	$f'(1)= -45$	$f'(3)= 675$
Sign	+ + + + + +	- - - - - - - - -	- - - - - - - - -	+ + + + + +
Shape	**Increasing**	**Decreasing**	**Decreasing**	**Increasing**
Intervals	$\boldsymbol{x < -2}$	$\boldsymbol{-2 < x < 0}$	$\boldsymbol{0 < x < 2}$	$\boldsymbol{x > 2}$

- **Step 4:**



Local Maximum at $x = -2$, Maximum = $f(-2) = 3(-2)^5 - 20(-2)^3 = 64$; or **Max (-2 , 64)**

Local Minimum at $x = 2$, Minimum = $f(2) = 3(2)^5 - 20(2)^3 = -64$; or **Min (2 , -64)**

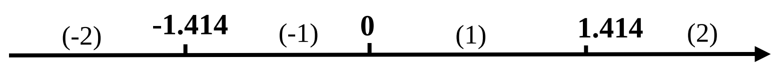
Increasing or $f'(x) > 0$ in: $x < -2$ and $x > 2$

Decreasing or $f'(x) < 0$ in: $-2 < x < 0$ and $0 < x < 2$, or $-2 < x < 2$

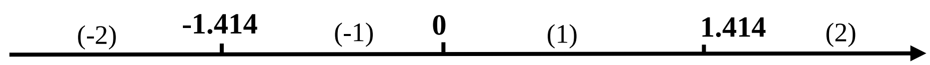
Example 2, continue

II) Using the Second Derivative:

- **Step 5:** Locate the inflection points by making $f''(x) = 0$:
  $f''(x) = 60x^3 - 120x$
  $f''(x) = 0$ then $60x(x^2 - 2) = 0$.
  Solve for x and you will find $\boldsymbol{x = 0}$, $\boldsymbol{x} = \pm\sqrt{2} = \pm 1.414$
- **Step 6:** Intervals & test points



- **Step 7:**

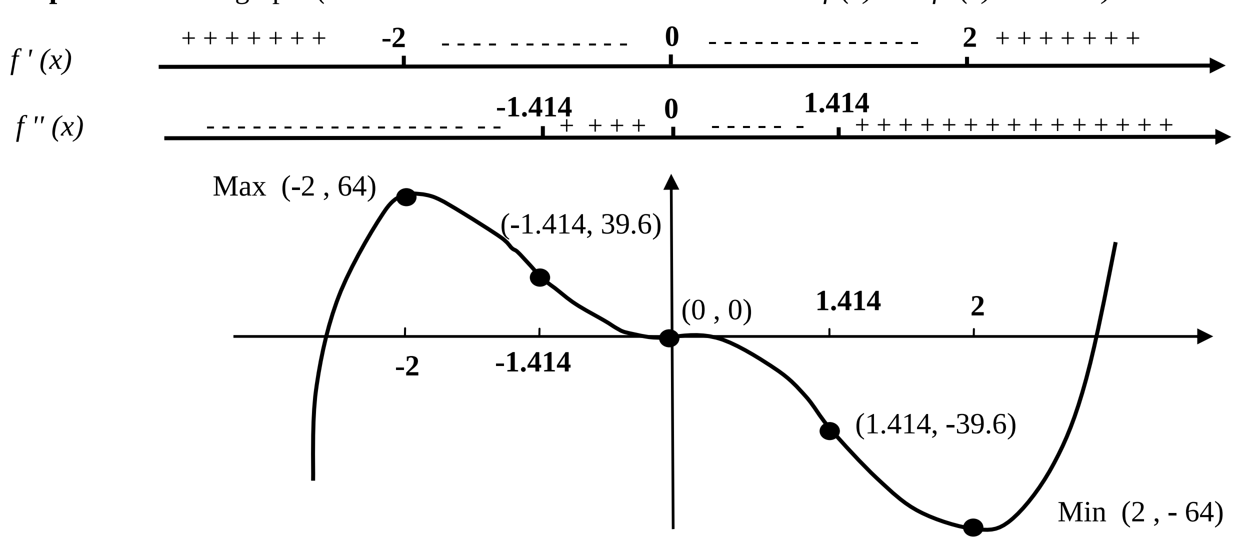


$f''(x) = 60x^3 - 120x$	$f''(-2)= -$	$f''(-1)= +$	$f''(1)= -$	$f''(2)= +$
Sign	- - - - - - - -	+ + + + + +	- - - - - - - -	+ + + + + +
Shape	**Concave Down**	**Concave Up**	**Concave Down**	**Concave Up**
Intervals	$\boldsymbol{x < -1.414}$	$\boldsymbol{-1.414 < x < 0}$	$\boldsymbol{0 < x < 1.414}$	$\boldsymbol{x > 1.414}$

- **Step 8:** Summarize all results in the following table:

Increasing in the intervals:	$\boldsymbol{x < -2}$ and $\boldsymbol{x > 2}$
Decreasing in the intervals:	$\boldsymbol{-2 < x < 2}$
Local Max. points and Max values:	Max. at $x = -2$, **Max (-2 , 64)**
Local Min. points and Min values:	Min. at $x = 2$, **Min (2 , -64)**
Inflection points at:	**(-1.414 , 39.6) , (0 , 0) , (-1.414 , -39.6)**
Concave Up in the intervals:	$\boldsymbol{-1.414 < x < 0}$ and $\boldsymbol{x > 1.414}$
Concave Down in the intervals:	$\boldsymbol{x < -1.414}$ and $\boldsymbol{0 < x < 1.414}$

- **Step 9:** Sketch the graph: *(Make sure the scale is consistent between f'(x) and f''(x) intervals)*



The following are extra examples, analyze them using the 9 steps , then check your final answers**:**

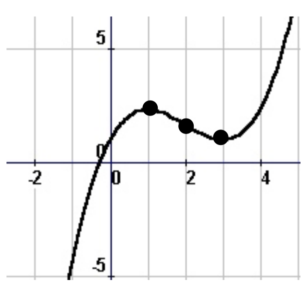
Example 3: $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x + 1$ **Example 4:** $f(x) = x^4 - 2x^2$

Example 5: $f(x) = x^4 - 4x^3$ **Example 6:** $f(x) = -3x^5 + 5x^3$

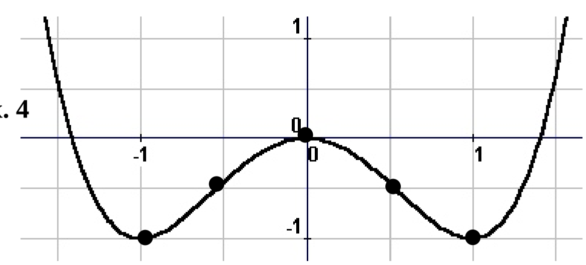
	Example3	**Example 4**
Increasing in the intervals:	$x < 1$ and $x > 3$	$-1 < x < 0$ and $x > 1$
Decreasing in the intervals:	$1 < x < 3$	$x < -1$ and $0 < x < 1$
Local Max. points and Max values:	Max. at $x = 1$, Max (1 , 7/3)	Max. at $x = 0$, Max (0 , 0)
Local Min. points and Min values:	Min. at $x = 3$, Min (3 , 1)	Min. at $x = -1,1$; Min (-1,-1) & (1,-1)
Inflection points at:	(2 , 5/3)	Approx. (-0.58 , -0.56) , (0.58 , -0.56)
Concave Up in the intervals:	$x > 2$	$x < -0.58$ and $x > 0.58$
Concave Down in the intervals:	$x < 2$	$-0.58 < x < 0.58$

	Example5	**Example 6**
Increasing in the intervals:	$x > 3$	$-1 < x < 1$
Decreasing in the intervals:	$x < 3$	$x < -1$ and $x > 1$
Local Max. points and Max values:	No local Max.	Max. at $x = 1$, Max (1 , 2)
Local Min. points and Min values:	Min. at $x = 3$, Min (3 , -27)	Min. at $x = -1$; Min (-1,-2)
Inflection points at:	(0 , 0) and (2 , -16)	Approx. (-0.707, -1.24), (0.707, 1.24), (0,0)
Concave Up in the intervals:	$x < 0$ and $x > 2$	$x < -0.707$ and $0 < x < 0.707$
Concave Down in the intervals:	$0 < x < 2$	$-0.707 < x < 0$ and $x > 0.707$

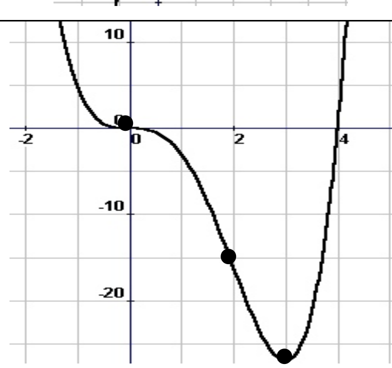
Ex. 3



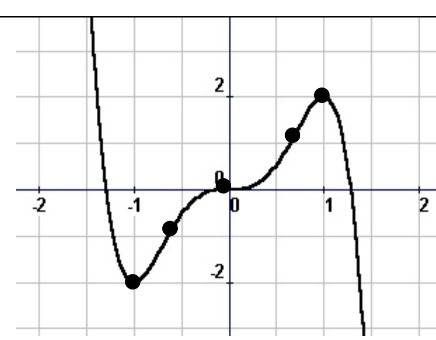
Ex. 4



Ex. 5



Ex. 6



The following are the graphs for problem **in page 180** in the book. Analyze each problem using the 9 steps, create the summary tables and sketch the graphs. Your summary tables can be verified from the graphs.

11) $f(x) = x^2 - 5x + 3$

13) $f(x) = 2x^3 + 3x^2 - 36x + 5$

16) $f(x) = 3x^4 - 4x^3 + 6$

17) $f(x) = x^4 - 8x^2 + 5$

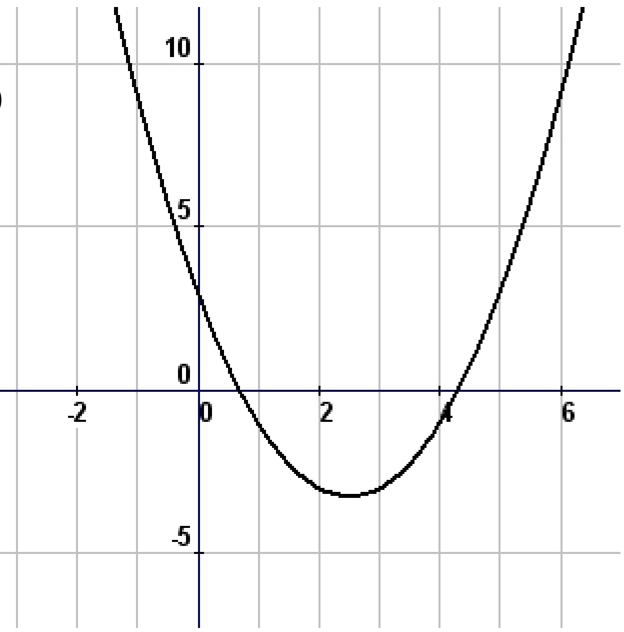
18) $y = x^4 - 4x^3 + 10$

20) $f(x) = 3x^5 - 5x^3$

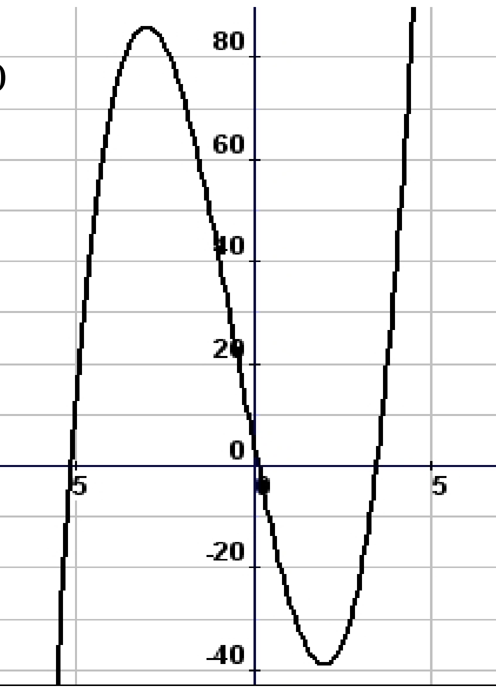
Extra 1: $f(x) = x^3 + 6x^2 + 9x - 1$

Extra 2: $f(x) = 20x^3 - 3x^5$

11)



13)



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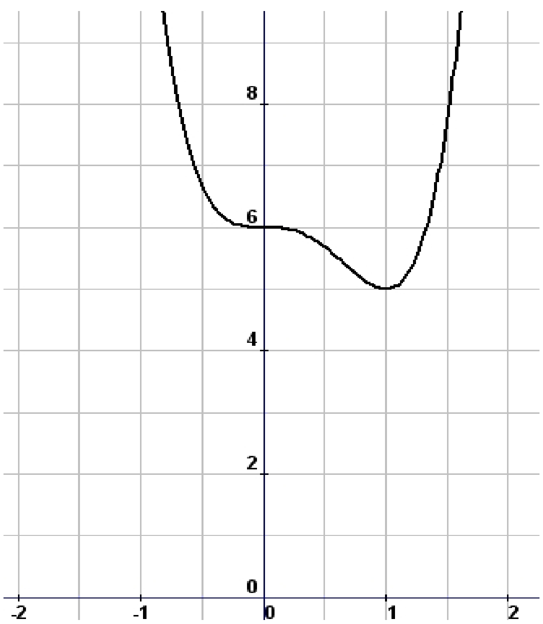

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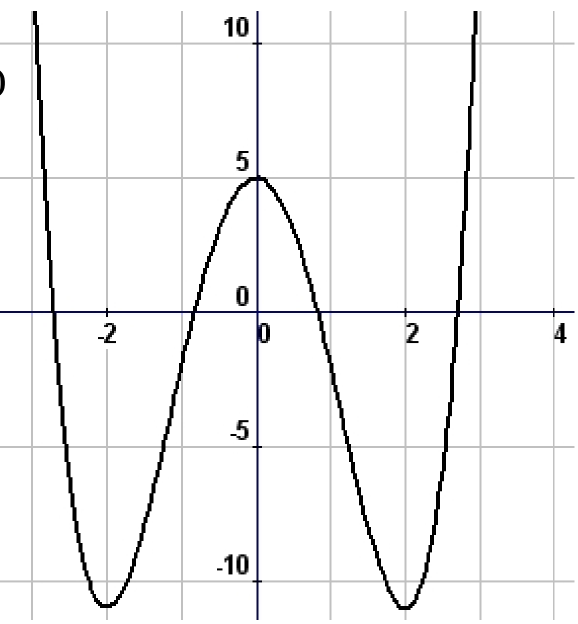

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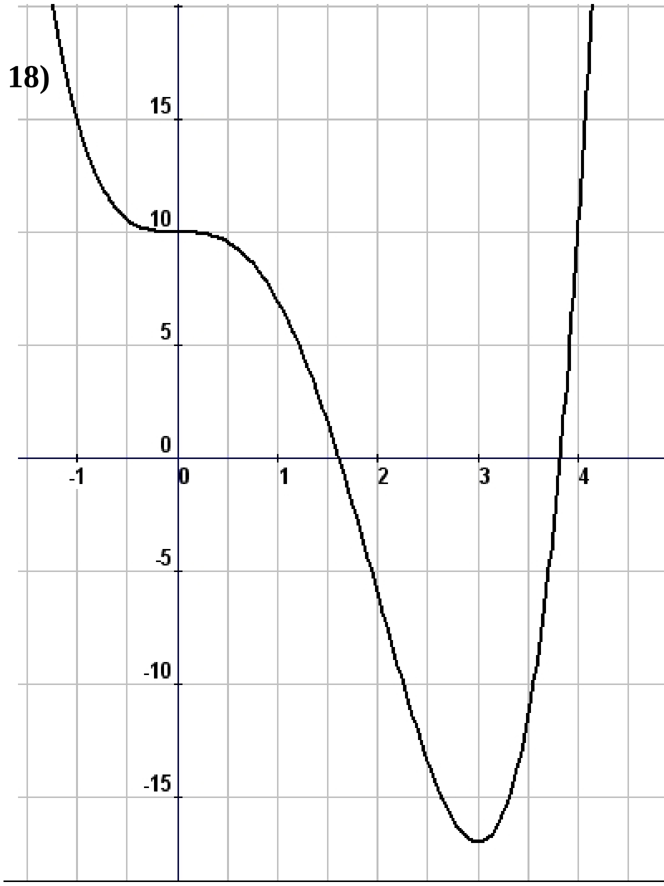

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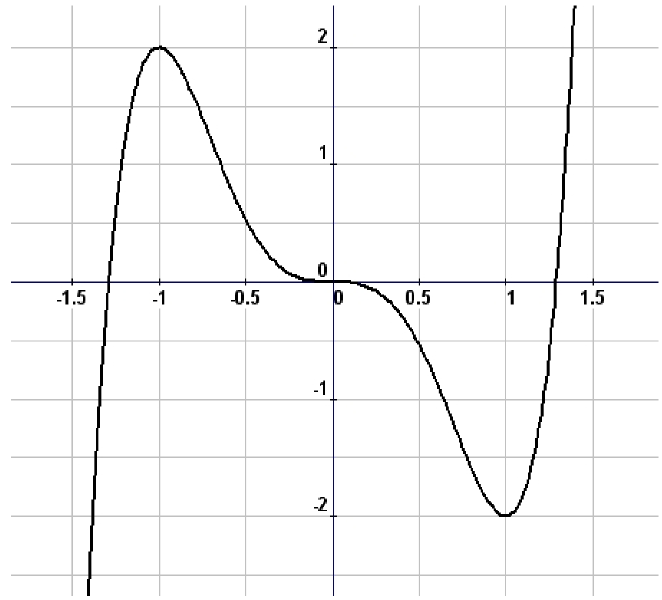
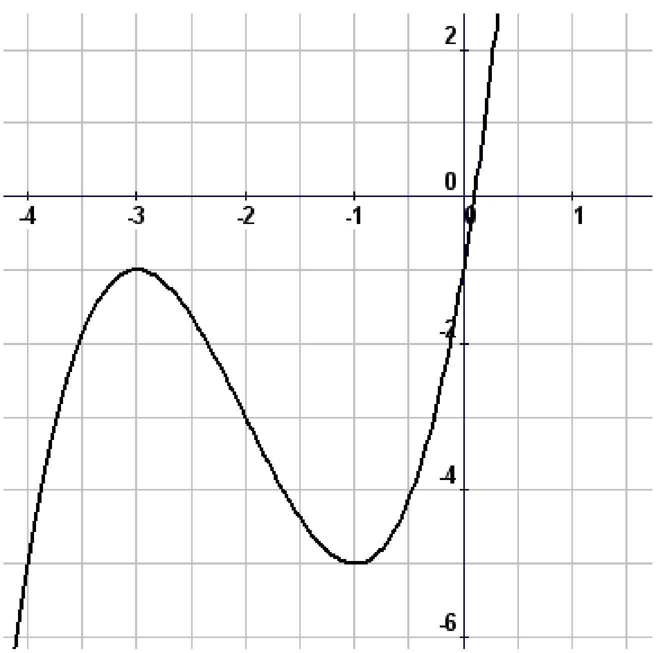


Extra 1



Extra 2